Unit 1 - Chapter 1
Mathematical logic

Chapter Structure

1.0 Objectives

Logic rules are used to give definite meaning to any mathematical statements.
Many algorithms and proofs use logical expressions such as:

> “IF p THEN q”
> or
> “If p1 AND p2, THEN q1 OR q2”

Logic rules are used to distinguish between valid and invalid mathematical arguments, i.e. to know the cases in which these expressions are TRUE or FALSE, that is, to know the “truth value” of such expressions.
We discuss the study of discrete mathematics with an introduction to logic in this chapter.
Logic has numerous applications to the computer science field such as, design of computer circuits, the construction of computer programs, the verification of the correctness of programs, and in many other ways.

1.1 Propositions and logical operations

Lets n begins with an introduction to the basic building blocks of logic—propositions.

1.1.1 Proposition

A proposition (or statement) is a declarative statement which is true or false, but not both. Let’s understand this by taking a few examples.

Consider following statements to find out whether they are propositions are not.

1)	Is it cold outside?	This is not a preposition, as it’s answer can have both the value true and false. So If a statement has a question mark it is not considered as preposition.
2)	Sun is bright.	This is a preposition , as this statement can either be true or false but not both.
3)	2 + 2 = 5	This is a preposition , as this statement can either be true or false but not both.
4)	Read this carefully.	This is not a preposition, as this is not a declarative sentence. It is a command.
5)	5 * X = 25	This is not a preposition, as its validity is dependent on the value of variable X . Hence it’s answer can have both the value true and false.

- We use letters to denote propositional variables (or statement variables).
- Variables that represent propositions are just as letters are used to denote numerical variables. The conventional letters used for propositional variables are p, q, r, s, ...
- The truth value of a proposition is **true**, denoted by **T**, if it is a true proposition, and
- The truth value of a proposition is **false**, denoted by **F**, if it is a false proposition.
- Propositional logic was first developed systematically by the Greek philosopher Aristotle more than 2300 years ago. It deals with propositional calculus.

Exercise:

Are these propositions?Give proper reasoning.

1. The sun is shining.
2. The sum of two prime numbers is even.
3. 3+4=7
4. It rained in Austin, TX, on October 30, 1999.

5. x+y > 10
6. Is it raining?
7. Come to class!
8. n is a prime number.
9. The moon is made of green cheese.

<u>Compound Statements or Compound Propositions:</u>

- Generally mathematical statements are constructed by combining one or more propositions.
- New propositions, called compound propositions, are formed from existing propositions using logical operators.

The fundamental property of a compound proposition is that its truth value is completely determined by the truth values of its subpropositions together with the way in which they are connected to form the compound propositions.

Many propositions are composite, that is, composed of subpropositions and various connectives discussed subsequently. Such composite propositions are called compound statements or compound propositions.
A proposition is said to be primitive if it cannot be broken down into simpler propositions, that is, if it is not composite.
For example, the below propositions
(i) Ice floats in water.
(ii) 2+2=6
are primitive propositions.

On the other hand, the following two propositions are composite:
(i) “Roses are red and violets are blue.” and
(ii) “John is smart or he studies every night.”

Let P(p, q, . . .) denote an expression constructed from logical variables p, q, . . ., which take on the value TRUE (T) or FALSE (F), and the logical connectives $\wedge$, $\vee$, and $\sim$ (and others discussed subsequently). Such an expression P(p, q, . . .) will be called a proposition. The letters p,q,r.... denotes propositional variables.
We can write the above compound statements as;
(i) p: Roses are red
q: violets are blue
thus, Compound statement is “p AND q”

(ii) p: John is smart

q: he studies every night
thus, Compound statement is “p OR q”

1.1.2 Basic Logical Operations

In this section we are discussing the three basic logical operations as follow:

1. Conjunction (AND) , symbolically ‘∧’
2. Disjunction (OR), symbolically ‘∨’
3. Negation (NOT), symbolically ‘¬’ or ‘~’

<u>Conjunction (AND) , symbolically ‘∧’</u>

Definition :

Let p and q be propositions. The conjunction of p and q, denoted by **p ∧ q**, is the proposition read as “p and q.” The conjunction p ∧ q is true when both p and q are true and is false otherwise.

Truth Table of p ∧ q:

Since p ∧ q is a proposition it has a truth value, and this truth value depends only on the truth values of p and q. Note that, *p ∧ q is true only when both p and q are true.*
Observe the following truth table for conjunction operation:

p	q	p ∧ q (p AND q)
T	T	T
T	F	F
F	T	F
F	F	F

Table: Conjunction / AND logical operation

Example :

Consider the following proposition and find out that given statement is having output true/false:

"Mumbai is the capital of India and 1 + 1 = 2."

Solution:
In this example let's assume,

p: Mumbai is capital of India
q: 1 + 1 = 2

Where we can get proposition p is false and q is true.
As per our truth table, it will give us "p AND q" is false, shown in below table:

p	q	p ∧ q (p AND q)
F	T	F

Disjunction (OR), symbolically '∨ '

Definition:

Let p and q be propositions. The disjunction of p and q, denoted by p ∨ q, is the proposition "p or q." The disjunction p ∨ q is false when both p and q are false and is true otherwise.

Truth Table of p ∨ q:

Since p ∨ q is a proposition it has a truth value, and this truth value depends only on the truth values of p and q. Note that, *p ∨ q is false only when both p and q are false.*
Observe the following truth table for disjunction operation:

p	q	p ∨ q (p OR q)
T	T	T
T	F	T

F	**T**	**T**
F	**F**	**F**

Table: Disjunction / OR logical operation

Example :

Consider the following proposition and find out that given statement is having output true/false:

"Mumbai is the capital of India and 1 + 1 = 2."

Solution:
In this example let's assume,

p: Mumbai is capital of India
q: 1 + 1 = 2

Where we can get proposition p is false and q is true.
As per our truth table, it will give us "p OR q" is false, shown in below table:

p	**q**	**p ∨ q (p OR q)**
F	**T**	**T**

Negation (NOT), symbolically '¬' or '~'

Definition :

Let p be a proposition.The negation of p, denoted by ¬p (also denoted by ~p or $\underline{p}$), is the statement "It is not the case that p." The proposition ¬p is read "not p."
The truth value of the negation of p, ¬p, is the opposite of the truth value of p.

Truth Table of ¬p:

The truth value of the negation of p is always the opposite of the truth value of p. It is a unary operator means it requires only one operand(proposition) to perform this operation.
Observe the following truth table for negation operation:

p	**¬p (NOT p)**
T	**F**
F	**T**

Table: Negation/NOT logical operation

Example :

Consider the following proposition and find out that given statement is having output true/false:

"Delhi is the capital of India."

Solution:
In this example let's assume,

p: Delhi is the capital of India

Then the negation of given proposition p is shown in below table:
It can be read as,
¬p : It is not the case that Delhi is the capital of India. OR
¬p : Delhi is **not** the capital of India. OR
¬p : It is not true that Delhi is the capital of India. OR
¬p : It is false that Delhi is the capital of India.

p	**¬p (NOT p)**
T	**F**

Tautologies And Contradictions :

Definition:
A proposition P (p, q, . . .) is called a **tautology** if it contains only T in the last column of its truth table or, in other words, if it is True for any truth values of its variables.
Example:

p	**~p**	**p v ~p**
T	**F**	**T**
F	**T**	**T**

Definition:
A proposition P (p, q, . . .) is called a **contradiction** if it contains only F in the last column of its truth table or, in other words, if it is false for any truth values of its variables

Example:

p	**~p**	**p ^**

		~p
T	F	F
F	T	F

Exercise:

1. Let p be "It is hot" and let q be "It is raining". Give a simple verbal sentence which describes each of the following statements:

   (a) ~p (b) p ∧ q (c) p ∨ q (d) q ∨~p.

2. Find the truth table of ~p ∧ q.
3. Verify that the proposition p ∨~(p ∧ q) is a tautology.
4. Find the truth tables for.

   (a) p ∨~q (b) ~p ∧~q.

5. Verify that the proposition (p ∧ q)∧~(p ∨ q) is a contradiction

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1.2 Conditional Statements

Many mathematical statements are of the form "If p then q." Such statements are called conditional statements and are denoted by p → q
There are some other common ways to express this conditional statement:

"if p, then q"
"p implies q"
"if p, q"
"p only if q"
"p is sufficient for q"
"a sufficient condition for q is p"
"q if p"
"q whenever p"
"q when p"
"q is necessary for p"
"a necessary condition for p is q"
"q follows from p"
"q unless ¬p

Definition:

Let p and q be propositions. The conditional statement p → q is the proposition "if p, then q." The conditional statement p → q is false when p is true and q is false, and true otherwise.

In the conditional statement p → q, p is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

Truth Table:

Note that the conditional p → q is false only when the first part p is true and the second part q is false.Accordingly, when p is false, the conditional p → q is true regardless of the truth value of q.

A conditional statement is also called **an implication**.

Observe the following truth table for conditional operation:

p	**q**	**p → q (if p, then q)**
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Table: conditional operation

Example:

From the implication **p → q** for each of the following:

(i) p:I am girl q: I will dance

can be written as , "If I am a girl, then I will dance".

(ii) p:It is summer q: 2+3= 5

written as , "If it is summer , then 2+ 3=5."

<u>Biconditionals :</u>

Another common statement is of the form "p if and only if q." Such statements are called biconditional statements and are denoted by p ↔ q.

Biconditional statements are also called **bi-implications**.

There are some other common ways to express p ↔ q:

- "p is necessary and sufficient for q"
- "if p then q, and conversely"
- "p iff q."

Definition:

Let p and q be propositions. The biconditional statement p ↔ q is the proposition "p if and only if q."

The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

Truth Table:

Note that the biconditional $p \leftrightarrow q$ is true whenever p and q have the same truth values and false otherwise.

Observe the following truth table for conditional operation:

p	q	$p \leftrightarrow q$ (p if and only if q)
T	T	T
T	F	F
F	T	F
F	F	T

Table: conditional operation

Example:
From the bi-implication $p \leftrightarrow q$ for each of the following:
p: A polygon is a triangle.
q: A polygon has exactly 3 sides.
$p \leftrightarrow q$: "A polygon is a triangle if and only if it has exactly 3 sides."

1.3 Methods of Proof

Method of mathematical proof is an argument we give logically to validate a mathematical statement. In order to validate a statement, we consider two things: A **statement** and **Logical operators**. A statement is either true or false but not both. Logical operators are AND, OR, NOT, If then, and If and only if. We also use quantifiers like for all(∀) and there exists(∃). We apply operators on the statement to check the correctness of it.

Let's discuss this in more detail:methods you can use to write proofs.

Methods of proof are as follow:

1.3.1 Direct Proof

Any problem solving method has both discovery and proof as integral parts. When you think you have discovered that a certain statement is true, try to figure out why it is true. If you succeed, you will know that your assumption is correct. Even if you fail, the process of trying will give you insight into the nature of the problem and may lead to the discovery that the the statement is false.

Few assumptions which we are going to consider in this section. In this text we assume a familiarity with the laws of basic algebra.

We also use the three properties of equality: For all objects A, B, and C,

➔ (1) A = A,
  (2) if A = B then B = A, and
  (3) if A = B and B = C, then A = C.
➔ In addition, we assume that there is no integer between 0 and 1 and that the set of all integers is closed under addition, subtraction, and multiplication. This means that sums, differences, and products of integers are integers.
➔ Of course, most quotients of integers are not integers. For example, 3 ÷ 2, which equals 3/2, is not an integer, and 3 ÷ 0 is not even a number.

Steps: Method of Direct Proof

1. Express the statement to be proved in the form "∀x ∈ D, if P(x) then Q(x)."
2. Start the proof by supposing x is a particular but arbitrarily chosen element of D for which the hypothesis P(x) is true. (This step is often abbreviated "Suppose x ∈ D and P(x).")
3. Show that the conclusion Q(x) is true by using definitions, previously established results, and the rules for logical inference.

A direct proof of theorem:

Theorem: The sum of any two even integers is even.
Proof: Suppose m and n are particular but arbitrarily chosen even integers. **We must show that m + n is even.** By definition of even, we can say that:

m = 2*r and
n = 2*s

for some integers r and s.

Then

m + n = 2r + 2s	by substitution
= 2(r + s)	by factoring out a 2.

Let t = r + s. (Note that t is an integer because it is a sum of integers.) Hence

m + n = 2t where t is an integer.

It follows by **definition of even** that m + n is even. Hence proved.

Theorem:
The sum of any two odd integers is even.

Proof:
Suppose m and n are particular but arbitrarily chosen odd integers.

We must show that m + n is even.

By definition of odd, we can say that:

m = 2*r + 1 and
n = 2*s + 1

for some integers r and s.

Then

m + n = (2r +1) + (2s + 1)	by substitution
= 2r + 2s + 2	by associative and commutative laws of addition
= 2(r + s + 1)	by factoring out a 2.

Let t = r + s + 1. (Note that t is an integer because it is a sum of integers.) Hence

m + n = 2t where t is an integer.

It follows by **definition of even** that m + n is even. Hence proved.

Exercise:

Prove the following statements using direct method of proof:

1) For all integers m and n, if m is odd and n is even, then m + n is odd.
2) The product of two odd numbers is odd.

1.3.1.1 Counterexample

A counterexample is an example that disproves a universal ("for all ∀ ") statement. Obtaining counterexamples is a very important part of mathematics. As we do require critical attitude toward claims.If you find a counterexample which shows that the given logic is false, that's good because progress comes not only through doing the right thing, but also by correcting your mistakes.

Suppose you have a quantified statement: "All x's satisfy property P": $\forall x P(x)$.

The second quantified statement says: "There is an x which does not satisfy property P".
In other words, to prove that "All x's satisfy property P" is false, you must find an x which does not satisfy property P.

By basic logic, $P \rightarrow Q$ is 1 false when P is true and Q is false. Therefore: To give a counterexample to a conditional statement $P \rightarrow Q$, find a case where P is true but Q is false.

Give a counterexample to the statement : If n is an integer and n^2 is divisible by 4, then n is divisible by 4.
Proof: To give a counterexample, we have to find an integer n such n^2 is divisible by 4, but n is not divisible by 4 the "if" part must be true, but the "then" part must be false. Consider n = 6. Then $n^2 = 6^2$ $= 36$ 36 is divisible by 4, but n = 6 is not divisible by 4. Thus, n = 6 is a counterexample to the statement.

Show that the following statement is false: There is a positive integer n such that n2 + 3n + 2 is prime.
Proof: Proving that the given statement is false is equivalent to proving its negation is true. The negation is For all positive integers n, $n^2 + 3n + 2$ is not prime. Because the negation is universal, it is proved by generalizing from the generic particular. Suppose n is any particular but arbitrarily chosen positive integer. We will show that $n^2 + 3n + 2$ is not prime. We can factor $n^2 + 3n + 2$ to obtain $n^2 + 3n + 2 = (n + 1)(n +2)$. We also note that n + 1 and n + 2 are integers (because they are sums of integers) and that both $n + 1 > 1$ and $n + 2 > 1$ (because $n \geq 1$). Thus $n^2 + 3n + 2$ is a product of two integers each greater than 1, and so $n^2 + 3n + 2$ is not prime.

Exercise:
Prove following using counterexample method of proof:

1) For all integers m and n, if 2m + n is odd then m and n are both odd.
2) For all integers n, if n is odd then $\frac{n-1}{2}$ is odd.

1.3.2 Indirect Proof

In a direct proof you start with the hypothesis of a statement and make one deduction after another until you reach the conclusion. Indirect proofs are more roundabout or winding.

1.3.2.1 Contradiction

One kind of indirect proof, argument by ***contradiction***, is based on the fact that either a statement is true or it is false but not both.

So if you can show that the assumption that a given statement is not true leads logically to a contradiction then that assumption must be false: and, hence, the given statement must be true.

The point of departure for a proof by contradiction is the supposition that the statement to be proved is false. The goal is to reason to a contradiction. Thus proof by contradiction has the following outline:

Steps: Method of Proof by Contradiction

1. Suppose the statement to be proved is false. That is, suppose that the negation of the statement is true.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement to be proved is true.

Theorem
There is no greatest integer.

Proof:

We take the negation of the theorem and suppose it to be true.
That is, suppose there is a greatest integer N.
We must deduce a contradiction.
Then $N \geq n$ for every integer n.

Let $M = N + 1$.

Now M is an integer since it is a sum of integers.

Also $M > N$ since $M = N + 1$.

Thus M is an integer that is greater than N.

So N is the greatest integer and N is not the greatest integer, which is a contradiction.
This contradiction shows that the supposition is false and, hence, that the theorem is true.

Theorem
There is no integer that is both even and odd.

Proof:

We take the negation of the theorem and suppose it to be true.

That is, suppose there is at least one integer n that is both even and odd.

We must deduce a contradiction.

By definition of even, n = 2a for some integer a, and by definition of odd, n = 2b + 1 for some integer b.

Consequently,

$2a = 2b + 1$ by equating the two expressions for n

$2a - 2b = 1$

$2(a - b) = 1$

$a - b = 1/2$ by algebra.

Now since a and b are integers, the difference a − b must also be an integer. But a − b = 1/2, and 1/2 is not an integer.

Thus a − b is an integer and a − b is not an integer, which is a contradiction. This contradiction shows that the supposition is false and, hence, that the theorem is true.

Exercise:

Prove following by contradiction method of proof:

1) The sum of any rational number and any irrational number is irrational.
2) The square root of any irrational number is irrational.
3) For all integers n, if n2 is even then n is even.

1.3.2.2 Contraposition

A second form of indirect argument, argument by contraposition, is based on the logical equivalence between a statement and its contrapositive. To prove a statement by contraposition, you take the contrapositive of the statement, prove the contrapositive by a direct proof, and conclude that the original statement is true.

Method of Proof by Contraposition

1. Express the statement to be proved in the form ∀ x in D, if P(x) then Q(x).
2. Rewrite this statement in the contrapositive form ∀ x in D, if Q(x) is false then

P(x) is false.

3. Prove the contrapositive by a direct proof.

   a. Suppose x is a particular but arbitrarily chosen element of D such that Q(x) is false.

   b. Show that P(x) is false.

Proposition

For all integers n, if n^2 is even then n is even.

Proof :

Suppose n is any odd integer.

We must show that n^2 is odd.

By definition of odd,

n = 2k + 1 for some integer k.

By substitution and algebra,

$$
\begin{aligned}
n^2 &= (2k + 1)^2 \\
&= 4k^2 + 4k + 1 \\
&= 2(2k^2 + 2k) + 1.
\end{aligned}
$$

But $2k^2 + 2k$ is an integer because products and sums of integers are integers.

So $n^2 = 2\cdot$(an integer) + 1, and thus, by definition of odd, n^2 is odd .

For any integers a and b, $a + b \geq 15$ implies that $a \geq 8$ or $b \geq 8$

Proof:

We'll prove the contrapositive of this statement.

We must show that for any integers a and b, $a < 8$ and $b < 8$ implies that $a + b < 15$.

By above statement ,

a and b are integers such that $a < 8$ and $b < 8$.

This implies that $a \leq 7$ and $b \leq 7$.

By substitution and algebra we get,

$a + b \leq 14$.

But this implies that $a + b < 15$.

Exercise:

Prove following by contrapositive method of proof:

1) If a product of two positive real numbers is greater than 100, then at least one of the numbers is greater than 10.
2) For all integers m and n, if mn is even then m is even or n is even.

1.3.2.3 Two Classical Theorems

This section contains proofs of two of the most famous theorems in mathematics: that √2 is irrational and that there are infinitely many prime numbers. Both proofs are examples of indirect arguments.

Theorem : Irrationality of √2 √2 is irrational.
Proof: We take the negation and suppose it to be true. That is, suppose √2 is rational. Then there are integers m and n with no common factors such that $\sqrt{2} = \frac{m}{n}$ by dividing m and n by any common factors if necessary We must derive a contradiction. Squaring both sides of equation gives $2 = m^2/n^2$ $m^2 = 2n^2$ This implies that m^2 is even by definition of even. It follows that m is even . m = 2k for some integer k. $m^2 = (2k)^2$ by substituting $= 4k^2$ $= 2n^2$ Dividing both sides of the right-most equation by 2 gives $n^2 = 2k^2$

Consequently, n^2 is even, and so n is even .

But we also know that m is even.
Hence both m and n have a common factor of 2.
But this contradicts the supposition that m and n have no common factors.
Hence the supposition is false and so the theorem is true.

Proposition: 1 + 3√2 is irrational.
Proof:

Suppose 1 + 3√2 is rational.

We must derive a contradiction.
Then by definition of rational,

$1 + 3\sqrt{2} = \frac{a}{b}$ for some integers a and b with $b \neq 0$.

It follows that

$3\sqrt{2} = \frac{a}{b} - 1$ by subtracting 1 from both sides

$= \frac{a}{b} - \frac{b}{b}$ by substitution

$= \frac{a-b}{b}$

$\sqrt{2} = \frac{a-b}{3b}$ by dividing both sides by 3.

But a − b and 3b are integers since a and b are integers and differences and products of
integers are integers, and $3b \neq 0$ by the zero product property.

Hence √2 is a quotient of the two integers a − b and 3b with 3b = 0, and so √2 is rational by definition of rational.

This contradicts the fact that √2 is irrational. This contradiction shows that the supposition is false.
Hence 1 + 3√2 is irrational.

Unit 1- Chapter 2
Logic

Chapter Structure

2.0 Objective

Proofs in mathematics are valid arguments that establish the truth of mathematical statements. By an argument, we mean a sequence of statements that end with a conclusion. By valid, we mean that the conclusion, or final statement of the argument, must follow from the truth of the preceding statements, or premises, of the argument.
That is, an argument is valid if and only if it is impossible for all the premises to be true and the conclusion to be false.

2.1 Mathematical Induction

One of the most important techniques to prove many mathematical statements or formulae which cannot be easily derived by direct methods is sometimes derived by using the principle of mathematical induction.

Steps for induction method :
1. *Basic step of induction* : Show it is true for the first one , the smallest integral value of n (n = 1,2,3).
2. *Induction step*: If the statement is true for n=k , where k denotes any value of n , then it must be true for n = k + 1.
3. *Conclusion*: The statement is true for all integral values of n equal to or greater than that for which it was verified in Step 1

Prove that 1 + 2 + 3 + . . . + n = $\frac{n(n+1)}{2}$ for any integer n ≥ 1.

1. Show it is true for n=1

L.H.S. = n = 1

R.H.S. = $\frac{n(n+1)}{2} = \frac{1(1+1)}{2} = \frac{1(2)}{2} = 1$

Hence, L.H.S = R.H.S

2. Assume it is true for n=k that is,

$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$

3. Prove that statement is true for n = k+ 1 that is,

$1 + 2 + 3 + \ldots + k + (k + 1) = \frac{k+1(k+1+1)}{2}$.

We have ,

$1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$

By putting value of k into (k+1) equation we get,

$\frac{k(k+1)}{2} + (k + 1) = \frac{k+1(k+1+1)}{2}$.

Now we will start solving L.H.S. to check whether it is equal to R.H.S. or not.
Therefore,

L.H.S = 1+2+3+.........k+ (k + 1)

$= \frac{k(k+1)}{2} + (k+1)$

$= \frac{(k+1)(k+2)}{2}$

$= \frac{(k+1)(k+1+1)}{2}$

= R.H.S

3.Hence, by the principle of mathematical induction, P(n) is true for all n ∈ N.

Using the principle of mathematical induction, prove that

n(n + 1)(n + 5) is a multiple of 3 for all n ∈ N.

Proof:

Let P(n): n(n + 1)(n + 5) is a multiple of 3.

For n = 1, the given expression becomes (1 × 2 × 6) = 12, which is a multiple of 3.

So, the given statement is true for $n = 1$, i.e. P(1) is true.

Let P(k) be true. Then,

P(k): $k(k + 1)(k + 5)$ is a multiple of 3

$\Rightarrow K(k + 1)(k + 5) = 3m$ for some natural number m, ... (i)

Now, $(k + 1l)(k + 2)(k + 6) = (k + 1)(k + 2)k + 6(k + 1)(k + 2)$

$= k(k + 1)(k + 2) + 6(k + 1)(k + 2)$

$= k(k + 1)(k + 5 – 3) + 6(k + 1)(k + 2)$

$= k(k + 1)(k + 5) – 3k(k + 1) + 6(k + 1)(k + 2)$

$= k(k + 1)(k + 5) + 3(k + 1)(k +4)$ [on simplification]

$= 3m + 3(k + 1)(k + 4)$ [using (i)]

$= 3[m + (k + 1)(k + 4)]$, which is a multiple of 3

$\Rightarrow$ P(k + 1): $(k + 1)(k + 2)(k + 6)$ is a multiple of 3

$\Rightarrow$ P(k + 1) is true, whenever P(k) is true.

Thus, P(1) is true and P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of mathematical induction, P(n) is true for all $n \in N$.

Exercise:

Prove following by mathematical induction method of proof:

1) For all $n \geq 1$, $1 + 4 + 7 + \cdot \cdot \cdot + (3n - 2) = n(3n - 1) /2$
2) For n any positive integer, $6^n - 1$ is divisible by 5.

2.2 Mathematical Statements

- To deduce new statements from statements we already have, we use rules of inference which are templates for constructing valid arguments.
- Rules of inference are our basic tools for establishing the truth of statements.
- In this section we will look at arguments that involve only compound propositions.
- We will define what it means for an argument involving compound propositions to be valid.
- Then we will introduce a collection of rules of inference in propositional logic.
- After studying rules of inference in propositional logic, we will introduce rules of inference for quantified statements.
- We will describe how these rules of inference can be used to produce valid arguments.
- These rules of inference for statements involving existential and universal quantifiers play an important role in proofs in computer science and mathematics, although they are often used without being explicitly mentioned.
- Finally, we will show how rules of inference for propositions and for quantified statements can be combined.
- These combinations of rules of inference are often used together in complicated arguments.

Atomic and Molecular Statements

- A statement is any declarative sentence which is either true or false.
- A statement is atomic if it cannot be divided into smaller statements, otherwise it is called molecular.
- Propositional logic is the simplest form of logic. Here the only statements that are considered are propositions, which contain no variables.
- Because propositions contain no variables, they are either always true or always false. Examples of propositions:
  • $2 + 2 = 4$. (Always true).
  • $2 + 2 = 5$. (Always false).
- Examples of non-propositions:
  • $x + 2 = 4$. (May be true, may not be true; it depends on the value of x.)
  • $x \cdot 0 = 0$. (Always true, but it's still not a proposition because of the variable.)

Predicate logic

- Using only propositional logic, we can express a simple version of a famous argument:
  • Socrates is a man.
  • If Socrates is a man, then Socrates is mortal.
  • Therefore, Socrates is mortal.
- This is an application of the inference rule called modus ponens, which says that from p and $p \rightarrow q$ you can deduce q. The first two statements are axioms (meaning we are given them as true without proof), and the last is the conclusion of the argument.

- What if we encounter Socrates's infinitely more logical cousin Spocrates?
  We'd like to argue
  • Spocrates is a man.
  • If Spocrates is a man, then Spocrates is mortal.
  • Therefore, Spocrates is mortal.

Quantifiers

What we really want is to be able to say when H or P or Q is true for many different values of their arguments. This means we have to be able to talk about the truth or falsehood of statements that include variables. To do this,we bind the variables using quantifiers, which state whether the claim we are making applies to all values of the variable (universal quantification), or whether it may only apply to some (existential quantification).

Universal quantifier

- The universal quantifier $\forall$ called as "for all".
- It means that a statement must be true for all values of a variable within some universe of allowed values (which is often implicit).
- For example, "all humans are mortal" could be written as:
  $\forall x : \text{Human}(x) \rightarrow \text{Mortal}(x)$ and
- "if x is positive then x + 1 is positive" could be written as:
  $\forall x : x > 0 \rightarrow x + 1 > 0$.

- If you want to make the universe explicit, use set membership notation.
- An example would be $\forall x \in Z : x > 0 \rightarrow x + 1 > 0$.
- This is logically equivalent to writing,
  $\forall x : x \in Z \rightarrow (x > 0 \rightarrow x + 1 > 0)$ or

  to writing $\forall x : (x \in Z \wedge x > 0) \rightarrow x + 1 > 0$,

  but the short form makes it more clear that the intent of $x \in Z$ is to restrict the range of x.

- The statement $\forall x : P(x)$ is equivalent to a very large AND;
- for example, $\forall x \in N : P(x)$ could be rewritten (if you had an infinite amount of paper) as $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge \ldots$.
- Normal first-order logic doesn't allow infinite expressions like this, but it may help in visualizing what $\forall x : P(x)$ actually means.

- Another way of thinking about it is to imagine that x is supplied by some adversary and you are responsible for showing that P(x) is true; in this sense, the universal quantifier chooses the worst case value of x

Existential quantifier

- The existential quantifier ∃ called as "there exists". It means that a statement must be true for at least one value of the variable.
- So "some human is mortal" becomes $\exists x : Human(x) \wedge Mortal(x)$.
- Note that we use AND rather than implication here;

  the statement $\exists x : Human(x) \rightarrow Mortal(x)$

  makes the much weaker claim that "there is something x, such that if x is human, then x is mortal," which is true in any universe that contains an immortal purple penguin since it isn't human, Human(penguin) → Mortal(penguin) is true.

2.3 Logic and Problem Solving

We say that a conclusion is "logical" when it follows from what we already believe.
For example if we believe that light travels faster than sound, then it follows logically that we should see lightning before we hear thunder.
In everyday life, logical conclusions simply make explicit what is already implicit in one's beliefs.
Note that logical conclusions may or may not be true.

Our beliefs are really premises, statements from which we draw conclusions.
A conclusion that follows logically from the premises is VALID, but it is TRUE only if the premises are true.
Conversely, faulty/incorrect logic could lead to a true statement.
An **argument form**, or **argument** for short, is a sequence of statements. All statements but the last one are called **premises** or **hypotheses**. The final statement is called the **conclusion**, and is often preceded by a symbol " ∴".
An argument is **valid** if the conclusion is true whenever all the premises are true.
The validity of an argument can be tested through the use of the truth table by checking if the **critical rows**, i.e. the rows in which all premises are true, will correspond to the value "true" for the conclusion.
For example:

Major premise: If you study hard, then you will get an "A".
Minor premise: You have studied hard.

Conclusion: You will get an "A".

Steps to determine the validity of the argument:
– Identify the premises and conclusion of the argument
– Construct the truth table for all premises and the conclusion
– Find critical rows in which all the premises are true
– If the conclusion is true in all critical rows then the argument is valid, otherwise it is invalid

---

Example 1:
Check the validity of given :
Anyone who lives in the city Mulund, also lives on the island of Mumbai.
Ketan lives on the island of Mumbai.
Therefore, Ketan lives in the city Mulund.

Solution:
Invalid
EVEN IF the first statement is true, it does not say that everyone who lives on Mumbai, also lives in Mulund. There are other cities on Mumbai -- Ghatkopar, Sion , Churchgate etc. So, even if Ketan lives on Mumbai, we don't know if the conclusion is true.

Example 1:
Check the validity of given :
Anyone who lives in the city Mulund, also lives on the island of Mumbai.
Ketan lives in the city Mulund.
Therefore, Ketan lives on the island of Mumbai

Solution:
Valid.
We don't need to know anything about Ketan. If anyone lives in Mulund, he or she lives in Mumbai.
Ketan lives in Mulund, so he must live in Mumbai.
IF the premises are true, we are locked into the conclusion being true.

ANALYZING ARGUMENTS USING TRUTH TABLES

To analyze an argument with a truth table:

1. Represent each of the premises symbolically
2. Create a conditional statement, joining all the premises with and to form the antecedent, and using the conclusion as the consequent.

3. Create a truth table for that statement. If it is always true, then the argument is valid.
4. A row of the truth table in which all the premises are true is called a **critical row.**
   a. **If there is a critical row in which the conclusion is false, then the argument is invalid.**
   b. **If the conclusion in every critical row is true, then the argument form is valid.**

Example : Determine whether the following arguments are valid.

$$p \rightarrow q$$
$$q \rightarrow r$$
$$\therefore p \vee q \rightarrow r$$

Constructing a truth table, we have:

	p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \vee q$	$p \vee q \rightarrow r$
$\rightarrow$	T	T	T	T	T	T	T
	T	T	F	T	F	T	F
	T	F	T	F	T	T	T
	T	F	F	F	T	T	F
$\rightarrow$	F	T	T	T	T	T	T
	F	T	F	T	F	T	F
$\rightarrow$	F	F	T	T	T	F	T
$\rightarrow$	F	F	F	T	T	F	T

To help, we mark the critical rows. Notice that all critical rows have a true conclusion and thus the argument is valid.

2.4 Normal Forms

Suppose, **A (P1, P2, ... , Pn)** is a statements formula where **P1, P2, ..., P6** are the atomic variables if we consider all possible assignments of the truth value to **P1, P2, ..., Pn** and obtain the resulting truth values of the formula A then we get the truth table for **A**, such a truth table contains **2^6** rows.

The formula may have the truth value **T** for all possible assignments of the truth values to the variables **P1, P2, ..., Pn**.

In this case, **A** is called identically true or tautology.

If **A** has the truth value **T** for at least one combinations of truth values assigned to **P1, P2, ..., Pn** then **A** is called satisfiable.

The problem of determining in a finite number of steps, whether a given statement formula is a tautology or a contradiction or least satisfiable is known as a decision problem.

2.4.1 Types of Normal form:

1) Disjunctive Normal form

We may call **conjunction as product** and **disjunction as sum** .
A product of the variable and their negations in a formula is called an elementary product. Similarly, a sum of the variables and their negations is called as an elementary sum.

Some statements hold for elementary sums and product:

1. For any elementary product, the necessary condition is false is when it contains at least one pair of a factor in which one is the negation of the other.
2. For any elementary sum, the necessary condition is true when it contains at least one pair of factors in which one is the negations of the other.

2) Conjunctive normal form

The conjunctive normal form of a given formula is the one which contains the product of elementary sums (that formula is equivalent in the given formula).

Observations

1. The procedure for obtaining a formula in conjunctive normal form is quite similar to that of disjunctive normal form.
2. The conjunctive normal form is not unique.
3. A given formula will be identical if every elementary sum presents in its conjunctive normal form are identically true.
4. The 3 hold if every elementary sum present in the formula has at least two factors in which one is the negation of the other.

3) Principle Disjunctive normal form

- Suppose, **P** and **Q** are two variables. We construct all possible formulas which consist of a conjunction of **P** or in negation and conjunction of **Q** or its negation.
- Now of the formulas should contains both a variable and its negation.
- For two variables **P** and **Q** there is **2^2** such formula.
- These formulas are called minterms or Boolean conjunction of **p** and **Q** from the truth tables of theses minterms, it is clear that no two minterms are equivalent.
- Each minterm has the truth value **T** for exactly one combination of the truth value of the variables **P** and **Q**.
- For a given formula an equivalent formula consisting of a disjunction of **minterms** only is known as its principle disjunction normal form.
- Such a normal form is also said to be the sum-product canonical form.

4) Principle conjunctive normal form

- Given a number of variables maxterm of these variables is a formula which consists of disjunction in which each variable or its negations but not both appear only once.
- Observe that the **maxterm** are the duals of **minterms**.

- Therefore each of the **maxterm** has the truth value **F** for exactly one combination of the truth values of the variables.
- The **principle of conjunctive normal form** or the **product-sum canonical form**, the equivalent formula consists of only the conjunction of the **maxterms** only.

Things to remember:

- Remember that we also called "or" "disjunction" and "and" "conjunction".
- A clause that contains only $\vee$ is called a <u>disjunctive clause</u> and
- A clause that contains only $\wedge$ is called a <u>conjunctive clause</u>.
    - Negation is allowed, but only directly on variables.
    - $P \vee \neg q \vee r$: a disjunctive clause
    - $\neg p \wedge q \wedge \neg r$: a conjunctive clause
    - $\neg p \wedge \neg q \vee r$: neither
- If we put a bunch of disjunctive clauses together with $\wedge$, it is called <u>conjunctive normal form</u>.
  For example:
  $(p \vee r) \wedge (\neg q \vee \neg r) \wedge q$ is in conjunctive normal form.
- Similarly, putting conjunctive clauses together with $\vee$, it is called <u>disjunctive normal form</u>.
  For example:
  $(p \wedge \neg q \wedge r) \vee (\neg q \wedge \neg r)$ is in disjunctive normal form.

- More examples:
    - $(p \wedge q \wedge \neg r \wedge s) \vee (\neg q \wedge s) \vee (p \wedge s)$: It is in disjunctive normal form.
    - $(p \vee q \vee \neg r \vee s) \wedge (\neg q \vee s) \wedge \neg s$: It is in conjunctive normal form.
    - $(p \vee r) \wedge (q \wedge (p \vee \neg q))$:It is not in a normal form.

- We can turn any proposition into either normal form.
    - By using the definitions we can get rid of $\rightarrow$, $\leftrightarrow$, and $\oplus$.
    - Use DeMorgan's laws to move any $\neg$ in past parenthesis, so they sit on the variables.
    - Use double negation to get rid of any $\neg\neg$ that showed up.
    - Use the distributive rules to move things in/out of parenthesis as we need to.
- **For example, converting to conjunctive normal form:**

$$
\begin{aligned}
\neg((\neg p \to \neg q) \land \neg r) &\equiv \neg((\neg\neg p \lor \neg q) \land \neg r) && \text{[definition]}\\
&\equiv \neg((p \lor \neg q) \land \neg r) && \text{[double negation]}\\
&\equiv \neg(p \lor \neg q) \lor \neg\neg r && \text{[DeMorgan's]}\\
&\equiv \neg(p \lor \neg q) \lor r && \text{[double negation]}\\
&\equiv (\neg p \land \neg\neg q) \lor r && \text{[DeMorgan's]}\\
&\equiv (\neg p \land q) \lor r && \text{[double negation]}\\
&\equiv (\neg p \lor r) \land (q \lor r) && \text{[distributive]}
\end{aligned}
$$

- It was actually in disjunctive normal form in the second-last step.

Need of implementing normal form:

- May be easier to prove equivalence: to show $A \equiv B$, convert both to normal form, and then re-write one proof backwards.
- Maybe we simplify a lot: if we end up with $(p \lor \neg p \lor \cdots)$ terms, we know they are true.
- Proving theorems about all propositions: only have to handle boolean expressions in a normal form and that covers every proposition.

Example : Convert given statement in DNF i.e. Disjunctive Normal Form

$$
\begin{aligned}
(p \to q) \to (\neg r \land q) &\equiv \neg(p \to q) \lor (\neg r \land q) && \text{[definition]}\\
&\equiv \neg(\neg p \lor q) \lor (\neg r \land q) && \text{[definition]}\\
&\equiv (\neg\neg p \land \neg q) \lor (\neg r \land q) && \text{[DeMorgan's]}\\
&\equiv (p \land \neg q) \lor (\neg r \land q) && \text{[double negation]}
\end{aligned}
$$

$$
\begin{aligned}
&\equiv (p \land \neg q) \lor (\neg r \land q) && \text{[as above]}\\
&\equiv (p \lor (\neg r \land q)) \land (\neg q \lor (\neg r \land q)) && \text{[distributive]}\\
&\equiv (p \lor (\neg r \land q)) \land (\neg q \lor \neg r) \land (\neg q \lor q) && \text{[distributive]}\\
&\equiv (p \lor (\neg r \land q)) \land (\neg q \lor \neg r) \land \mathbf{T} && \text{[negation]}\\
&\equiv (p \lor (\neg r \land q)) \land (\neg q \lor \neg r) && \text{[identity]}\\
&\equiv (p \lor \neg r) \land (p \lor q) \land (\neg q \lor \neg r) && \text{[distributive]}
\end{aligned}
$$

• To convert a formula into a CNF.
– Open up the implications to get ORs.
– Get rid of double negations.
– Convert

$p \lor (q \land r)$ to

$(p \lor q) \land (p \lor r)$ [distributivity]

Example:

A → (B ∧ C)

≡ ¬ A ∨ (B ∧ C)

≡ (¬ A ∨ B) ∧ (¬ A ∨ C)

2.4 Exercise:

Laws of Propositional Logic:

S.No	Name of Laws	Primal Form	Dual Form
1	Idempotent Law	$p \vee p \equiv p$	$p \wedge p \equiv p$
2	Identity Law	$p \vee F \equiv p$	$p \wedge T \equiv p$
3	Dominant Law	$p \vee T \equiv T$	$p \wedge F \equiv F$
4	Complement Law	$p \vee \neg p \equiv T$	$p \wedge \neg p \equiv F$
5	Commutative Law	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
6	Associative Law	$p \vee (q \vee r) \equiv (p \vee q) \vee r$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
7	Distributive Law	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \wedge (p \wedge r)$
8	Absorption Law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
9	De Morgan's Law	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
10	Double Negation Law	$\neg(\neg p) \equiv p$	-

Solve the following:

1) Use mathematical induction to show that $1+2+3+\cdots+n=n(n+1)/2$ for all integers $n \geq 1$.
2) Use induction to prove that the following identity holds for all integers $n \geq 1$:
   $1+3+5+\cdots+(2n-1)=n^2$
3) Prove or disprove the following statement:
   $(p \vee q) \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$ are logically equivalent.
4) Determine whether the following arguments are valid or invalid:

   a)

   All triangles have angles.
   A circle has no angle.
   A circle is not a triangle.

   b)

   If you don't work hard then you won't succeed.
   You work hard.
   Therefore, you will succeed.

   c)

If you make an A on the midterm, you won't have to take the final.
Jose did not take the final.
Therefore, Jose made an A on the midterm.

5) Obtain distinctive forma (DNF) $(\sim P \vee \sim Q) \rightarrow (P \leftarrow\rightarrow Q)$.
6) Check the validity of following using truth table:
   Premise: If I go to the shop, then I'll buy new pen
   Premise: If I buy new pen, I'll buy a book to go with it
   Conclusion: If I got to the shop, I'll buy a book.

UNIT 2 - Chapter 3
SETS AND RELATIONS

Unit Structure

3.0 OBJECTIVES

After going through this unit, you will be able to:

- represent the relation
- classify the relation

3.1 INTRODUCTION

We get elements from a real system. These elements may be numbers, cities, items used to study a particular system etc. The collection of such well defined objects is called as a set. Depending up on the number of element of sets we classify sets. Sometimes we are interested in the relation between elements of two sets, the two sets may or may not be different.

3.2 RELATIONS

Partition of a Set: A partition of a non-empty set A is a collection P of its non-empty subsets A_i such that

Each element of A belongs to one of the sets in P.

If A_1 and A_2 are distinct elements of P, then $A_1 \cap A_2 = \phi$.

Cartesian Product of two sets:

If A and B are two non-empty sets, their Cartesian product $A \times B$ as the set of all ordered pairs (a,b) with $a \in A$ and $b \in B$.

Thus $A \times B = \{(a,b) / a \in A, b \in B\}$

Note that, for any finite, non empty sets A and B, $|A \times B| = |A||B|$

Relation:

Any subset of $A \times B$ is called as relation from A to B.

We will use capital letter like R, S etc. to denote the relations.

There are various methods of representing a relation.

3.2.1 METHOD OF ORDERED PAIRS

In this method we write all the pairs of the relation in a curly bracket.
For example A={1, 2, 3, 4, 5], B={a, b, c} and R is a relation from A to B, where R={(1,a), (2,a), (5,c)}. Here relation R is represented using the ordered pairs.

3.2.2 MATRIX OF RELATION

If A={a_1, a_2, a_3, a_4, …., a_m} and B={b_1, b_2, b_3, b_4, …., b_n} are finite sets containing m and n elements, respectively, and R is a relation from A to B, we represent R by m×n matrix $M_R=[m_{ij}]$ which is defined by

$$m_{ij} = \begin{cases} 1 \; if \; (a_i, b_j) \in R \\ 0 \; if \; (a_i, b_j) \notin R \end{cases}$$

The matrix M_R is called the matrix of R.
For example A={1, 2, 3, 4, 5], B={a, b, c} and R is a relation from A to B, where R={(1,a), (2,a), (5,c)}. The matrix of relation R is given below.

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

3.2.3 DIGRAPH OF A RELATION

A relation from a set A to itself is called as relation on A.

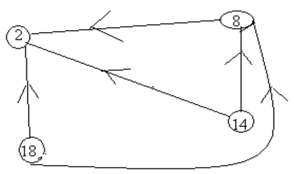
If A is a finite set and R is a relation on A, we can also represent R pictorially as follows.

Draw a small circle for each element of A and label the circle with the corresponding element of A. These circles are called vertices. Draw an arrow, called edge from a_i to vertex a_j if and only if $(a_i,a_j) \in R$. The resulting pictorial representation of R is called a directed graph or digraph of R.

For example: Let C={2,8,14,18}. Define a relation R on C by xRy if and only if $x - y > 5$.

In this case we get R as R={(8,2), (14,2), (18,2), (14,8), (18,8)}

The Digraph of R is shown below.



3.3 PROPERTIES OF RELATION

Once a relation is defined, we want to know its properties. These properties are useful in any applications. The relation with there properties are (i) Reflective, Irreflective Relation, (ii) Symmetric, Asymmetric and Antisymmetric Relation and (iii) Transitive Relation

Some relations satisfy one or more properties.

3.3.1 **REFLECTIVE, IRREFLECTIVE** RELATION

A relation R on a set A is <u>reflective</u> if $(a,a) \in R$ for all $a \in A$. (i.e. Every element is related to itself).

A relation R on a set A is <u>irreflective</u> if $(a,a) \notin R$ for every $a \in A$. (i.e. No element is related to itself).

3.3.2 SYMMETRIC, ASYMMETRIC AND ANTISYMMETRIC RELATION

A relation R on a set A is <u>symmetric</u>, if whenever $(a,b) \in R$ then $(b,a) \in R$. It then follows that R is not symmetric if we have a and $b \in A$ with $(a,b) \in R$ but $(b,a) \notin R$.

A relation R on a set A is <u>asymmetric</u>, if whenever $(a,b) \in R$ then $(b,a) \notin R$. It then follows that R is not asymmetric if we have some a and $b \in A$ with $(a,b) \in R$ and then $(b,a) \in R$.

A relation R on a set A is <u>antisymmetric</u>, if whenever $(a,b) \in R$ and $(b,a) \in R$ then $a=b$.
The contrapositive of this definition is that R is antisymmetric if whenever $a \neq b$ then $(a,b) \notin R$ or $(b,a) \notin R$. It follows that R is not antisymmetric if we have a and b in A, $a \neq b$ and with $(a,b) \in R$ and $(b,a) \in R$

3.3.3 **TRANSITIVE** RELATION

We say that a relation R on a set A is transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$.

A relation R on A is not transitive if there exist a, b and c in A so that $(a,b) \in R$ and $(b,c) \in R$ $(b,c) \notin R$.

<u>Note:</u> A relation R is transitive if and only if its matrix $M_R=\{m_{ij}\}$ has the property

$m_{ij}=1$ and $m_{jk}=1$ then $m_{ik}=1$

<u>Example 2.1:</u> Determine whether the relation R on a set A is reflective, irreflective, symmetric, asymmetric, antisymmetric or transitive. Give necessary explanation to your answer.

a) A=set of all positive integers, aRb iff $|a-b| \leq 2$.

b) Let A={1, 2, 3, 4} and R={ (1,1), (2,2), (3,3)}

c) A= set of all positive integers, aRb iff GCD (a,b)=1

<u>Solution</u>: a) A=set of all positive integers, aRb iff $|a-b| \leq 2$

(i) R is reflective because $|a-a| \leq 2 \ \forall a \in A$.

(ii) R is not irreflective because $|3-3| \leq 2$ for $3 \in A$

(iii) R is symmetric because for $|a-b| \leq 2$ implies $|b-a| \leq 2$ $\therefore aRb \rightarrow bRa$.

(iv) R is not asymmetric because for $|4-3| \leq 2$ we have $|3-4| \leq 2$ i.e. $4R3 \rightarrow 3R4$.

(v) R is not antisymmetric because 4R3 and 3R4 but $4 \neq 3$.

(vi) R is not transitive because 4R3, 3R1 but $4 \not R 1$ because $|4-1| \not\leq 2$

b) Let A={1, 2, 3, 4} and R={ (1,1), (2,2), (3,3)}

(i) R is not reflective because $(4,4) \notin R$

(ii) R is not irreflective because $(1,1) \notin R$.

(iii) R is symmetric because whenever $(a,b) \in R$ then (b,a) $) \in R$.

(iv) R is not asymmetric.

(v) R is antisymmetric.

(vi) R is transitive.

c) A= set of all positive integers, aRb iff GCD (a,b)=1.

In this case, we say that a and b are relatively prime.

(i) R is not reflective because $GCD(3,3) \neq 1$ it is 3 $\therefore (3,3) \notin R$.

(ii) R is not irreflective because GCD(1,1) =1.

(iii) R is symmetric because for GCD(a,b)=1 implies GCD(b,a)=1 $\therefore aRb \rightarrow bRa$.

(iv) R is not asymmetric because GCD(a,b)=1 then GCD(b,a)=1 $\therefore aRb \rightarrow bRa$.

(v) R is not antisymmetric because 4R3 and 3R4 but $4 \neq 3$.

(vi) R is not transitive because 4R3, 3R2 but $4 \not{R} 2$ because GCD(4,2) =2.

3.4 EQUIVALENCE RELATION

A relation R on A is called an **equivalence relation** if it is reflective, symmetric, and transitive.

<u>Example 2.2:</u> Let A be a set of all triangles in the plane and R be the relation on A defined as follows:

$R=\{(a,b) \in A \times A \mid a \text{ is congruent to } b\}$

Show that R is an equivalence relation.

<u>Solution:</u> <u>Step-1:</u> To show that R is reflective.

Since every triangle is congruent to itself we have $(a,a) \in R \ \forall a \in A$. Hence R is reflective.

<u>Step-2:</u> To show that R is symmetric.

If triangle a is congruent to triangle b, then triangle b is congruent to triangle a. $\forall a,b \in A$. Hence R is symmetric.

<u>Step-3:</u> To show that R is transitive.

If triangle a is congruent to triangle b and triangle b is congruent to triangle c then triangle a is congruent to triangle c. $\forall a,b,c \in A$. Hence R is transitive.

$\therefore$ R is equivalence relation.

2.4 Equivalence Relations
    2.4.1 Equivalence Classes
    2.4.2 Quotient Set

2.5 Partial Order Relation
    2.5.1 Hasse Diagram

3.4.1 EQUIVALENCE CLASSES

If R is an equivalence relation on A, then the set R(a) are called equivalence classes of R where $a \in A$.

Some authors denote R(a) by [a]

Example 2.3: Let S= Set of integers. Define relation R on A=S×S as

aRb if and only if $a \equiv b \pmod 2$. Read as "a is congruent to b mod 2."

Show that R is an equivalence relation. Determine the equivalence classes.

Solution: First, clearly $a \equiv a \pmod 2$. Thus R is reflective.

Second, if $a \equiv b \pmod 2$, then a and b yield the same remainder when divided by 2, so $b \equiv a \pmod 2$. R is symmetric.

Finally, suppose that $a \equiv b \pmod 2$ and $b \equiv c \pmod 2$. Then a, b, and c yield the same remainder when divided by 2. Thus, $a \equiv c \pmod 2$. Hence congruent mod 2 is an equivalence relation.

Equivalences classes are

$[0] = \{ ----, -4, -2, 0, 2, 4, ----\}$

$[1] = \{ ----, -3, -1, 1, 3, 5, ----\}$

3.4.2 QUOTIENT SET

If R is an equivalence relation on A, then set of all equivalence classes of R, is called as quotient set. It is denoted by A/R.

The set of all equivalence classes (quotient set) form partition of A.

Example 2.5: Let S= Set of integers. Define relation R on A=S×S as aRb if and only if a≡b (mod 5).Read as “a is congruent to b mod 5.”

Show that R is an equivalence relation. Determine the quotient set A/R .

Solution: First, clearly a≡a (mod 5). Thus R is reflective.

Second, if a≡b (mod 5), then a and b yield the same remainder when divided by 5, so b≡a (mod 5). R is symmetric.

Finally, suppose that a≡b (mod 5) and b≡c (mod 5). Then a, b, and c yield the same remainder when divided by 5. Thus, a≡c (mod 5). Hence congruent mod 5 is an equivalence relation.

Equivalences classes are

[0]= { ----,-10,-5, 0, 5, 10 , ----}

[1]= { ----,-9,-4, 1, 6, 11 , ----}

[2]= { ----,-8,-3, 2, 7, 12 , ----}

[3]= { ----,-7,-2, 3, 8, 13 , ----}

[4]= { ----,-6,-1, 4, 9, 14 , ----}

The quotient set A/R = {[0], [1], [2], [3],[4]}

Example 2.6: Let S={1, 2, 3, 4} and let A=S×S. Define the following relation R on A:

(a,b)R(a’,b’) if and only if a+b=a’+b’

i) Show that R is an equivalence relation.

ii) Compute A/R.

Solution: Step-1. To show that R is reflective.

Since a+b=a+b, we have (a,b)R(a,b). Hence R is reflective.

Step-2: To show that R is symmetric:

If a+b=a’+b’ then a’+b’=a+b i.e. (a,b)R(a’,b’) implies (a’,b’)R(a,b). Hence R is symmetric.

Step-3: To show that R is transitive:

If a+b=a’+b’ and a’+b’=a”+b” then a+b=a”+b” i.e. (a,b)R(a’,b’) and (a’,b’)R(a”,b”) implies (a,b)R(a”,b”). Hence R is transitive.

Hence R is equivalence relation.

Now R((1,1))={(1,1)} R((1,2))={(1,2), (2,1)};

R((1,3))={(1,3), (2,2), (3,1)} R((1,4))={(1,4), (2,3), (3,2), (4,1)}

R((2,4))={(2,4), (3,3), (4,2)} R((3,4))={(3,4), (4,3)}

R((4,4))={(4,4)}

$\therefore$ A/R={ {(1,1)}, {(1,2), (2,1)}, {(1,3), (2,2), (3,1)}, {(1,4), (2,3), (3,2), (4,1)},

{(2,4), (3,3), (4,2)}, {(3,4), (4,3)}, {(4,4)} }

3.5 OPERATIONS ON RELATION

Let R and S be relations from set A to a set B. Then, if we remember that R and S are simply subsets of A×B, we can use set operations on R and S.

• The complement of R, $\overline{R}$, is referred to as the **complementary relation**. It is of course, a relation from A to B that can be expressed simply in terms of R:

$$(a,b)\in \overline{R} \text{ b if and only if } (a,b)\notin R.$$

• We can also form the **intersection** R∩S and the **union** R∪S of the relations R and S. In relational terms, we see that (a,b)∈R∩S means that (a,b)∈R and (a,b)∈S. All our set theoretic operations can be used in this way to produce new relations.

• A different type of operation on a relation R from A to B is the formation of **inverse**, usually written R^{-1}. The relation R^{-1} is the relation from B to A (reverse order from R) defined by

$$(b,a)\in R^{-1} \text{ if and only if } (a,b)\in R.$$

Example 2.7: Let A={1,2,3,4} and B={a,b,c}.

Let R={(1,a), (1,b), (2,b), (2,c), (3,b), (4,a)} and S={(1,b), (3,b), (4,b)}

Compute (a) $\overline{R}$, (b) R∩S, (c) R∪S, (d) R^{-1}.

Solution: We first find

A×B={(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c), (4,a), (4,b), (4,c)}

(a) $\overline{R}$ = {(1,c), (2,a), (3,a),(3,c), (4,b), (4,c)}

(b) R∩S = {(1,b), (3,b)}

(c) R∪S = {(1,a), (1,b), (2,b), (2,c), (3,b), (4,a), (4,b)}

R^{-1} = {(a,1), (b,1), (b,2), (c,2), (b,3), (a,4)}

<u>Example 2.8</u>: Let A={1,2,3} and let R and S be the relations on A. Suppose that matrices of R and S are

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Give matrices of : (i) $R \cup S$ (ii) $R \cap S$ (iii) R^{-1} (iv) $\overline{S}$

<u>Solution:</u>

(i) $M_{R\cup S} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ (ii) $M_{R\cap S} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) $M_{R^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$ (iv) $M_{\overline{S}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

(iv) $M_{S^{--}} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3.6 PARTIALLY ORDERED SET (POSET)

A relation R on a set A is called a partial order if R is reflective, Antisymmetric and transitive. A set A together with partial order R is called partially ordered set or simply a poset and will denote it by (A,R).

Note: If there is no possibility of confusion about partial order, we may refer to the poset simply as A, rather than (A,R).

Example 2.9: Let A be a collection of subsets of a set S. The relation $\subseteq$ of set inclusion is a partial order on A, so $(A,\subseteq)$ is a poset.

Solution: (i) R is reflective

Since $A_1 \subseteq A_1 \quad \forall A_1 \in A \quad \therefore A_1 R A_1 \quad \therefore$ R is reflective.

(ii) R is antisymmetric.

If $A_1 \subseteq A_2$ and $A_2 \subseteq A_1$ then $A_1 = A_2$ $\therefore$ R is antisymmetric.

(iii) R is transitive.

If $A_1 \subseteq A_2$ and $A_2 \subseteq A_3$ then $A_1 \subseteq A_3$ $\therefore$ R is transitive.

$\therefore \subseteq$ is partial order relation.

$\therefore (A, \subseteq)$ is a poset.

3.6.1 HASSE DIAGRAM

The simplified diagram of the digraph of a partial order relation(R) is called as Hasse Diagram.

(i) Since partial order relation is always reflection, delete edges of the type (a,a) from the digraph of partial order relation.

(ii) Since partial order relation is always transitive, delete edges of the type (a,c) from the digraph of partial order relation if $(a,b) \in R$ and $(b,c) \in R$.

(iii) Do not show arrow along the edges of the digraph. Do not use circles to show vertices of the digraph.

(iv) Use some order. Write smaller elements on lower level.

The resulting diagram of a partial order , much simpler than its digraph, is called the Hasse diagram of the partial order of the poset.

Example 2.10 : Let A={1,2,3,4,12}. Consider the relation R as aRb iff ‘a divides b’. Show that R is partial order relation. Draw the Hasse diagram of the poset (A, R).

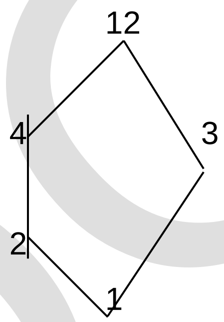
Solution: (i) R is reflective because every element divides itself i.e. $(a,a)\in R$ for every $a\in A$.

(i) R is antisymmetric because if a|b and b|a then a=b i.e. $(a,b)\in R$ and $(b,a)\in R$ implies a=b.

(ii) R is transitive because if a|b and b|c then a|c i.e. $(a,b)\in R$ and $(b,c)\in R$ implies $(a,c)\in R$.

Hence R is a partial order relation.

The Hasse diagram is shown below



Exercise 2:

(1) Let A={a,b,c,d} and let R be the relation on A that has the matrix

(i) $M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$ (ii) $M_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

Construct the diagraph of R , and list in-degrees and out-degrees of all vertices.

(2) Let A=R=Set of real numbers. For the following relation defined on A, find the domain and range for each,

(i) aRb iff $a^2+b^2=25$ (ii) aRb iff $2a+3b=6$

(3) Let A=set of real numbers . We define the following relation R on A.

$(x,y) \in R$ if and only if x and y satisfy the equation $\frac{x^2}{4}+\frac{y^2}{9}=1$.

Find R(x) and R(1).

(4) Let A={2, 3, 6, 12} and R and S be the following relations on A.

xRy iff x-y is divisible by 2

xSy iff x-y is divisible by 3.

Find (i) The matrix corresponding to R and S.

(ii) The matrices corresponding to $R \cap S$ and $R \cup S$.

(iii)The matrices corresponding to $\bar{R}$ and $\bar{S}^{-1}$.

(5) Let A={1,2,3,4} and let R={(1,2),(2,2),(3,4),(4,1)}. Is R symmetric, asymmetric or antisymmetric?

Unit 3 - Chapter 4

Graph

Chapter Overview:

4.0 Objective

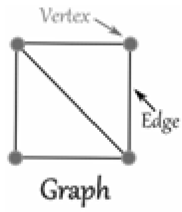
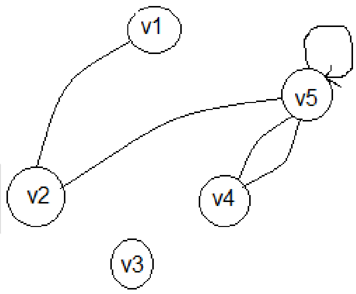
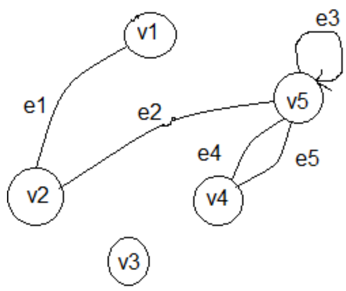
Graphs, directed graphs, play vital roles in mathematics and computer science.In mathematics, graph theory is the study of graphs, which can represent mathematical structures which are used to model pairwise relations between objects. Graph is a mathematical representation of a network and it describes the relationship between lines and points. A graph consists of some points (known as nodes) and lines (known as edges) between them. The length of the lines and position of the points do not matter. Each object in a graph is called a node.

4.1 Graph

Pictures like the dot and line drawing are called graphs. Graphs are made up of a collection of dots called vertices and lines connecting those dots called edges. When two vertices are connected by an edge, we say they are adjacent. The nice thing about looking at graphs instead of pictures of rivers, islands and bridges is that we now have a mathematical object to study. We have distilled the “important” parts of the bridge picture for the purposes of the problem. It does not matter how big the islands are, what the bridges are made out of, if the river contains alligators, etc. All that matters is which land masses are connected to which other land masses, and how many times.

Graph Definition.
A graph is an ordered pair G (V, E) consisting of a nonempty set V (called the vertices) and a set E (called the edges) of two-element subsets of V.

Definitions and Fundamental Concepts are as follows :

Definition	**Example**
Conceptually, a graph is formed by vertices and edges connecting the vertices. Example. Formally, a graph is a pair of sets (V, E), where V is the set of vertices and E is the set of edges, formed by pairs of vertices	
Often, we label the vertices with letters (for example: a, b, c, . . . or v1, v2, . . .) or numbers 1, 2, . . .	
Example. We have V = {v1, . . . , v5} for the vertices and E = {(v1, v2),(v2, v5),(v5, v5),(v5, v4),(v5, v4)} for the edges.	
Remark. The two edges (u, v) and (v, u) are the same. In other words, the pair is not ordered.	
Example. (Continuing from the previous example) We label the edges as follows: So E = {e1, . . . , e5}.	

Applications of Graph Theory

Graph theory has its applications in diverse fields of engineering −

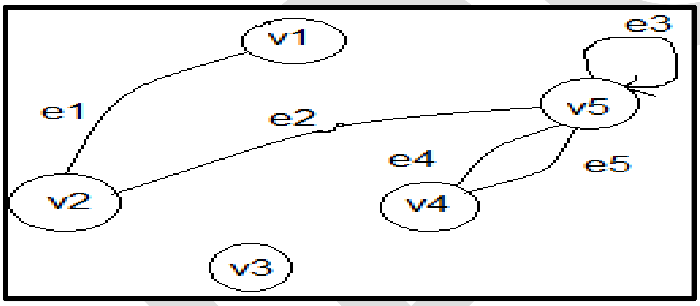
- **Electrical Engineering** − The concepts of graph theory is used extensively in designing circuit connections. The types or organization of connections are named as topologies. Some examples for topologies are star, bridge, series, and parallel topologies.
- **Computer Science** − Graph theory is used for the study of algorithms. For example,

- Kruskal's Algorithm
- Prim's Algorithm
- Dijkstra's Algorithm

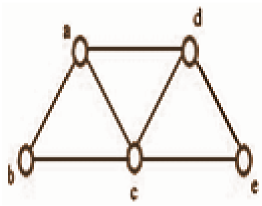
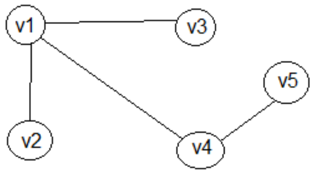
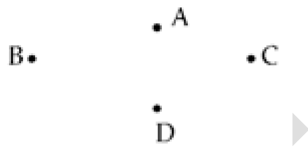
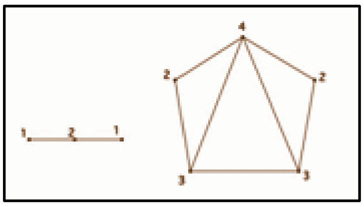
- **Computer Network** − The relationships among interconnected computers in the network follows the principles of graph theory.
- **Science** − The molecular structure and chemical structure of a substance, the DNA structure of an organism, etc., are represented by graphs.
- **Linguistics** − The parsing tree of a language and grammar of a language uses graphs.
- **General** − Routes between the cities can be represented using graphs. Depicting hierarchical ordered information such as family tree can be used as a special type of graph called tree.

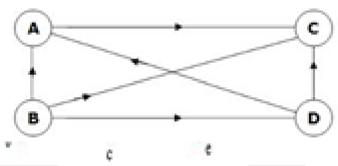
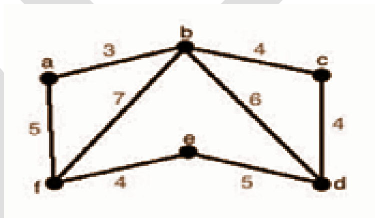
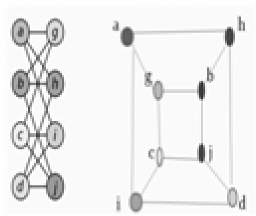
Different terminologies used in graph theory are as given :

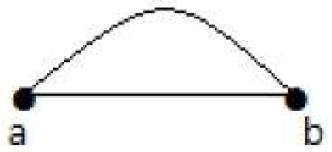
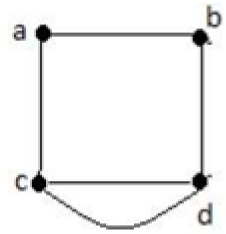
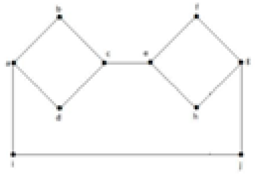
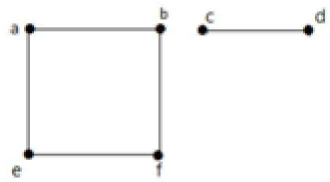
Consider the following example:



	Commonly used terms in graph	Definition	Example (considering above graph)
1	**End Vertices**	The two vertices u and v are end vertices of the edge (u, v).	v4 and v5 are end vertices of e5.
2	**Parallel Vertices**	Edges that have the same end vertices are parallel.	e4 and e5 are parallel.
3	**Loop**	An edge of the form (v, v) is a loop.	e3 is a loop.

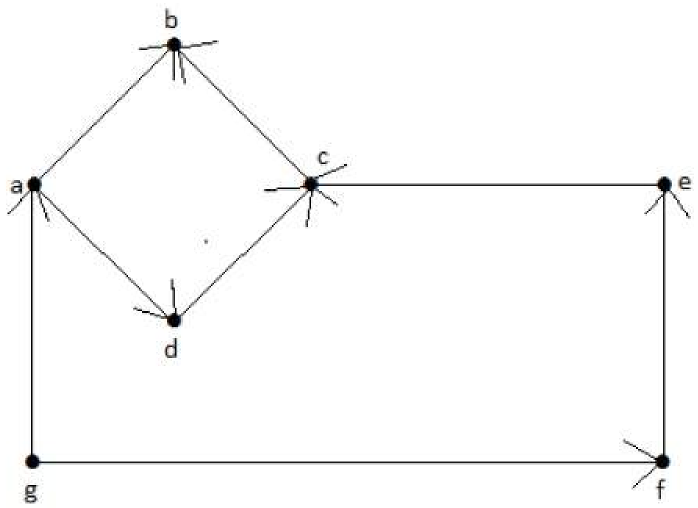
4	**Undirected Graph**	An undirected graph is one in which edges have no orientation.	
5	**Simple Graph**	A graph is simple if it has no parallel edges or loops. The given graph is not simple.	
6	**Empty Graph**	A graph with no edges (i.e. E is empty) is empty.	
7	**Null Graph**	A graph with no vertices (i.e. V and E are empty) is a null graph.	
8	**Trivial Graph**	A graph with only one vertex is trivial.	
9	**Adjacent Edges**	Edges are adjacent if they share a common end vertex.	e1 and e2 are adjacent.
10	**Adjacent Vertices**	Two vertices u and v are adjacent if they are connected by an edge, i.e., (u, v) is an edge.	v1 and v2 are adjacent
11	**Degree Of Vertex**	● The degree of the vertex v, written as d(v), is the number of edges with v as an end vertex. ● By convention, we count a loop twice and parallel edges contribute separately.	➔ The degree of v4 is 2. ➔ The degree of v5 is 5. Other Example: The degree of each vertex in this graph below is represented by the number. 

12	**Pendant Vertex**	A pendant vertex is a vertex whose degree is 1.	The degree of v1 is 1 so it is a pendant vertex.
13	**Pendent Edge**	An edge that has a pendant vertex as an end vertex is a pendant edge.	e1 is a pendant edge.
14	**Isolated Vertex**	An isolated vertex is a vertex whose degree is 0.	The degree of v3 is 0 so it is an isolated vertex.
15	**Complete Graph**	A simple graph is called a complete graph if each pair of distinct vertices is joined by an edge	
16	**Weighted Graph:**	A graph is a weighted graph if a number (weight) is assigned to each edge.Such weights might represent, for example, costs, lengths or capacities, etc. depending on the problem at hand . These graph are also called as a network	
17	**Isomorphic Graph:**	Two graphs are said to be isomorphic if there is one to one correspondence between their vertices and their edges such that incidences are preserved Properties preserved by isomorphism of graphs. • must have the same number of vertices • must have the same number of edges • must have the same number of vertices with degree k • for every proper subgraph g of one graph, there must be a proper subgraph of the other graph that is isomorphic of g	

18	**Parallel Edges**	In a graph, if a pair of vertices is connected by more than one edge, then those edges are called parallel edges.	
19	**Multi Graph**	A graph having parallel edges is known as a Multigraph.	
20	**Connected Graph**	A graph G is said to be connected if there exists a path between every pair of vertices. There should be at least one edge for every vertex in the graph. So that we can say that it is connected to some other vertex at the other side of the edge.	
20	**Disconnected Graph**	A graph G is disconnected, if it does not contain at least two connected vertices.	In this example, there are two independent components, a-b-f-e and c-d, which are not connected to each other. Hence this is a disconnected graph. 

Example 1

Take a look at the following directed graph. Vertex ‘a’ has two edges, ‘ad’ and ‘ab’, which are going outwards. Hence its outdegree is 2. Similarly, there is an edge ‘ga’, coming towards vertex ‘a’. Hence the indegree of ‘a’ is 1.

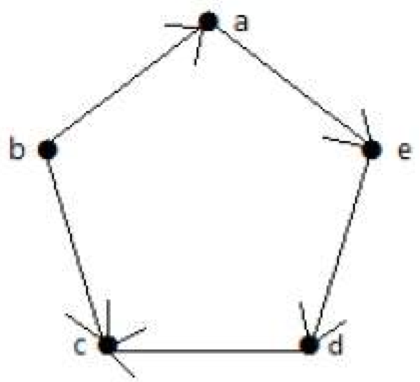


The indegree and outdegree of other vertices are shown in the following table –

Vertex	Indegree	Outdegree
a	1	2
b	2	0
c	2	1
d	1	1
e	1	1
f	1	1
g	0	2

Example 2

Take a look at the following directed graph. Vertex ‘a’ has an edge ‘ae’ going outwards from vertex ‘a’. Hence its outdegree is 1. Similarly, the graph has an edge ‘ba’ coming towards vertex ‘a’. Hence the indegree of ‘a’ is 1.

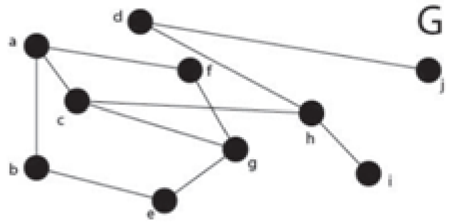


The indegree and outdegree of other vertices are shown in the following table –

Vertex	Indegree	Outdegree
a	1	1
b	0	2
c	2	0
d	1	1
e	1	1

Exercise :

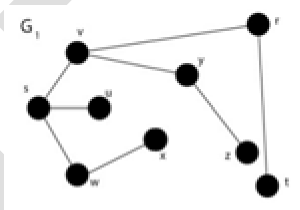
1. Consider following graph and answer the followed question:



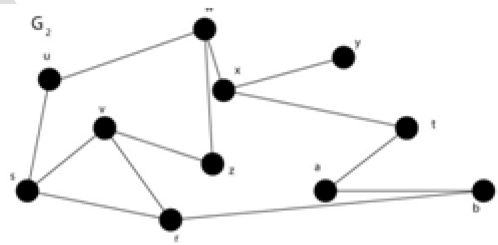
a) List the vertices in G
b) List the edges in G
c) List the edges incident with vertex e
d) List the vertices adjacent to g
e) Find a path from j to b
f) Find a cycle in G
g) Find the degree of each vertex and graph G.

2. Solve following for given graphs.

a) List the edges and vertices of the graph.
b) How many edges and vertices are there?
c) List the neighbours of the vertex v
d) How many edges are incident with s
e) Find a walk between s and t. Is your walk a path? Why or why not?
f) find a cycle



(i)



(ii)

4.2 Representation of Graph

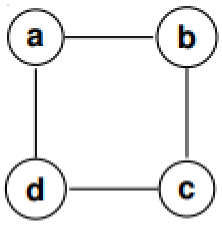
We can represent graphs in two ways :

1. As an Adjacency Matrix
2. As an Adjacency List
3. As an Incidence Matrix

Let's look at each of them in detail.

Adjacency Matrix of an Undirected Graph

Let the adjacency matrix = [a_{ij}] of a graph G is the n × n (n = |V |) zero-one matrix, where a_{ij} = 1 if {v_i , v_j} is an edge of G, and is 0 otherwise.



$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

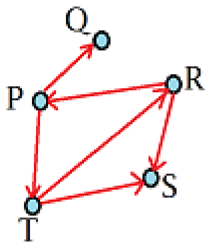
The adjacency matrix A of a graph G does depend on the ordering of the vertices of G, that is, a different ordering of the vertices yields a different adjacency matrix. However, any two such adjacency matrices are closely related in that one can be obtained from the other by simply interchanging rows and columns.

On the other hand,the adjacency matrix does not depend on the order in which the edges (pairs of vertics) are input into the computer.

4.2.1 Adjacency Matrix of a Directed Graph

For a directed graph, if there is a directed edge between two vertices, then the value is considered to be 1, else it is considered to be 0.

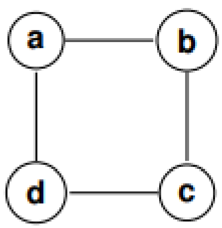
For example, in the following graph, there is a directed edge between the vertices P and Q. Therefore, this relationship would have a value of 1 in the matrix. However, the value for the edge Q-->P would be 0, as it is not a directed edge.



$$\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

4.2.2 Adjacency Lists

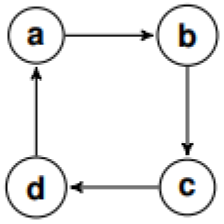
- Can be used to represent a graph with no multiple edges
- A table with 1 row per vertex, listing its adjacent vertices.



Vertex	Adjacent Vertex
a	b, d
b	a, c
c	b, d
d	a, c

Directed Adjacency Lists

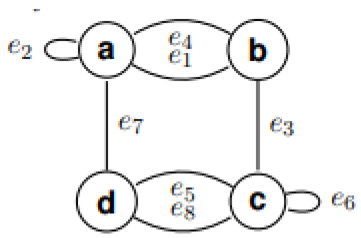
- 1 row per vertex, listing the terminal vertices of each edge incident from that vertex.



Initial Vertex	Terminal Vertices
a	b
b	c
c	d
d	a

5.2.3 Incidence Matrices

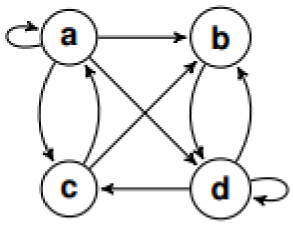
Let G = (V, E) be an undirected graph with $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$ where $m_{ij} = 1$ if e_j is incident with v_i, and is 0 otherwise.



$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{matrix} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

Example 1:

Use an adjacency list and adjacency matrix to represent the given graph



Solution:

Initial Vertex	Terminal Vertex
a	a, b, c, d
b	d
c	a, b
d	b, c, d

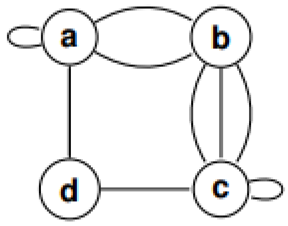
$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Example 2:

Draw an undirected graph represented by the given adjacency matrix.

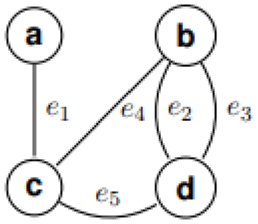
$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Solution:



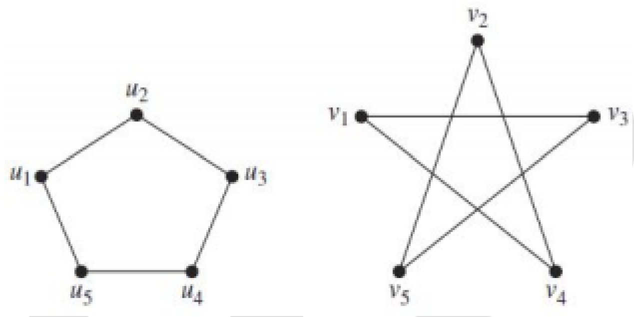
Exercise:

1) Use an incidence matrix to represent the graph.

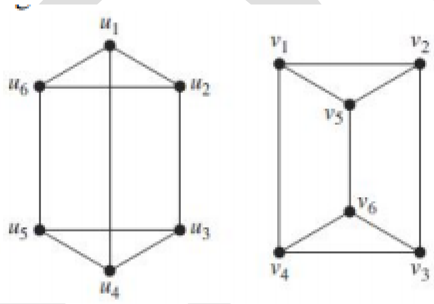


2) Determine whether the pair of graphs is isomorphic.

a)



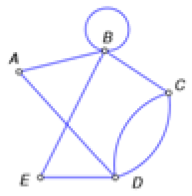
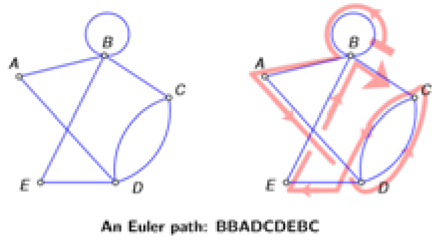
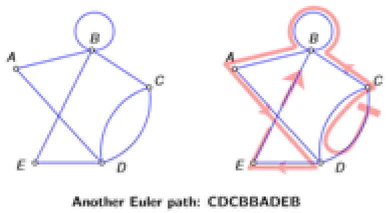
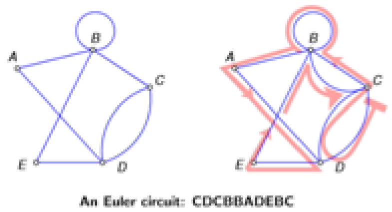
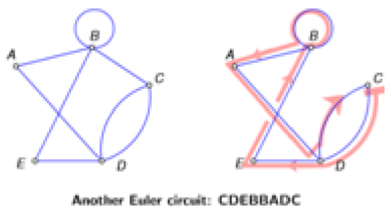
b)



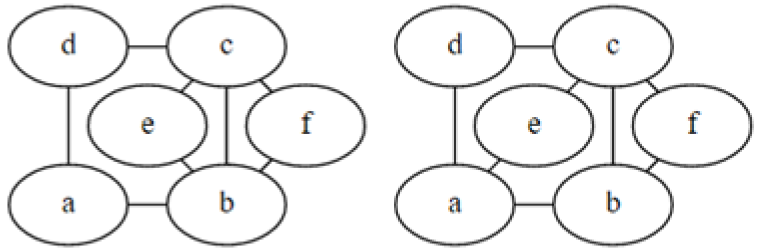
4.3 Euler paths and Circuits

An Euler path is a path that uses every edge of a graph exactly once.
An Euler circuit is a circuit that uses every edge of a graph exactly once.

- An Euler path starts and ends at different vertices.
- An Euler circuit starts and ends at the same vertex.

Euler Paths and Euler Circuits	
An Euler path: BBADCDEBC	
Another Euler path: CDCBBADEB	
An Euler circuit: CDCBBADEBC	
Another Euler circuit: CDEBBADC	

- An <u>Euler circuit</u> (or <u>Eulerian circuit</u>) in a graph G is a simple circuit that contains every edge of G.
    - Reminder: a simple circuit doesn't use the same edge more than once.
    - So, a circuit around the graph passing by every edge exactly once.
    - We will allow simple or multigraphs for any of the Euler stuff.
- In the modern world: you want to walk around the mall without missing any stores, or wasting time walking the same hall again.
- For example, the first graph has an Euler circuit, but the second doesn't.



Note: you're allowed to use the same <u>vertex</u> multiple times, just not the same edge.

- An <u>Euler path</u> (or <u>Eulerian path</u>) in a graph G is a simple path that contains every edge of G.
    - The same as an Euler circuit, but we don't have to end up back at the beginning.
    - The other graph above does have an Euler path.
- **Theorem:** A graph with an Eulerian circuit must be connected, and each vertex has even degree.

  Proof: If it's not connected, there's no way to create a circuit.

  When the Eulerian circuit arrives at an edge, it must also leave. This visits two edges on the vertex. When it returns to its starting point, it has visited an even number of edges at each vertex.
- **Theorem:** A connected graph with even degree at each vertex has an Eulerian circuit.

  Proof: We will show that a circuit exists by actually building it for a graph with $|V|=n$.

  For $n=2$, the graph must be two vertices connected by two edges. It has an Euler circuit.

  For $n>2$, pick a vertex v as a starting point. Pick an arbitrary edge leaving v. Continue to pick edges and walk around the graph until you return to v. We know we'll never get stuck since every vertex has even degree: if we walk in, then there's a way to walk out.

  This process forms part of our circuit. Let Ev be the set of edges visited in our initial loop.

  Consider the graph with edges $E-Ev$ and whatever vertices still have an edge adjacent. Each vertex in this graph has even degree (since we removed an even number from each) and it has less than n edges. By strong induction, we can find an Euler circuit for each connected component of this graph.

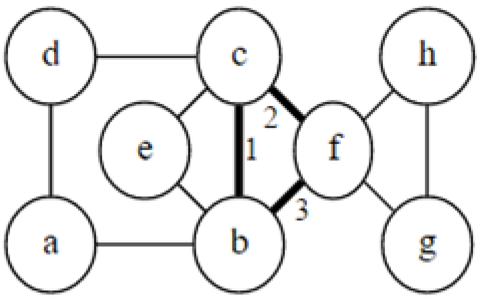
  Since our graph was connected originally, each of these sub-circuits shares a vertex with out Ev walk. We can join these together at the shared vertex to form a circuit of all edges in G.

Algorithm for Constructing an Euler Circuit

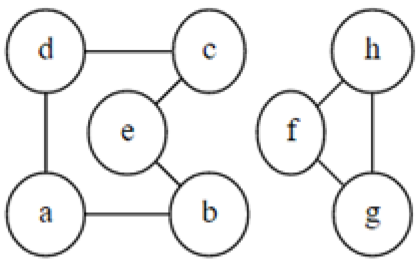
Algorithm $euler(G$: connected multigraph with all vertices of even degree)

$circuit$ = a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex
$H = G$ with the edges of this circuit and all isolated vertices removed
while H has edges **do**
  $subcircuit$ = a circuit in H beginning at a vertex in G that also is an endpoint of an edge of $circuit$
  $H = H$ with edges of $subcircuit$ and all isolated vertices removed
  $circuit = circuit$ with $subcircuit$ inserted at the appropriate vertex
end while
return $circuit\{circuit$ is an Euler circuit$\}$

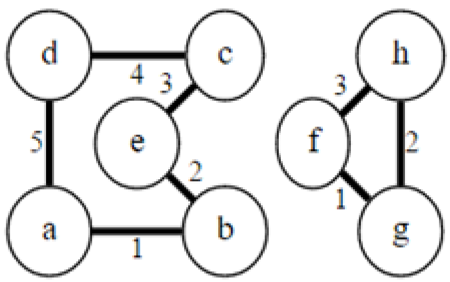
- An example will help. Suppose we have the graph below start at *b* and find the initial walk highlighted.



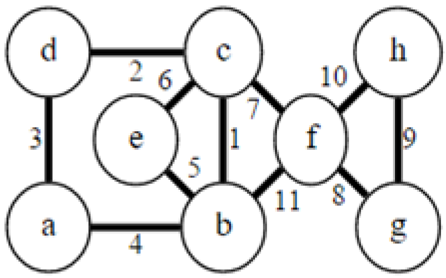
  - That leaves us with this two-component graph to apply the inductive hypothesis to:



  - So we find a Euler circuit in each component:

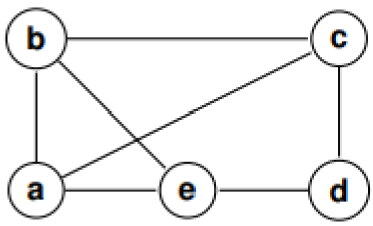


  - We combine to form a Euler circuit of the original by following one of the component-circuits whenever we can:



- **Corollary:** A graph has an Eulerian circuit if and only if it is connected and all of its vertices have even degree.
- **Corollary:** A graph has an Eulerian path but no Eulerian circuit if and only if it has exactly two vertices with odd degree.
  Proof: If we add an edge between the two odd-degree vertices, the graph will have an Eulerian circuit. If we remove the edge, then what remains is an Eulerian path.
  Suppose a graph with a different number of odd-degree vertices has an Eulerian path. Add an edge between the two ends of the path. This is a graph with an odd-degree vertex and a Euler circuit. As the above theorem shows, this is a contradiction.▮

Example 1:
Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists.



Solution:
By using the Euler path theorem we can say that,

➔ This graph does not have an Euler circuit because we have four vertices of odd degree.

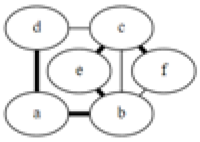
➔ This graph does not have an Euler path because we have four vertices of odd degree.

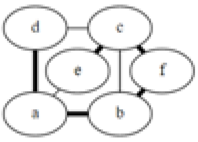
4.4 Hamiltonian Paths and Circuits

- The Euler circuits and paths wanted to use every edge exactly once.
- It seems obvious to then ask: can we make a circuit of a graph using every vertex exactly once?

- ○ Such a circuit is a <u>Hamilton circuit</u> or <u>Hamiltonian circuit</u>.
- ○ Similarly, a path through each vertex that doesn't end where it started is a <u>Hamilton path</u>.

- It seems like finding a Hamilton circuit (or conditions for one) should be more-or-less as easy as a Euler circuit.
  - Unfortunately, it's much harder.
- For example, the two graphs above have Hamilton paths but not circuits:



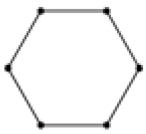


- Somehow, it feels like if there “enough” edges, then we should be able to find a Hamiltonian cycle. The following two theorem give us some good-enough conditions.
- **Theorem:** (Ore's Theorem) In a graph with $n \geq 3$ vertices, if for each pair of vertices either $\deg(u)+\deg(v) \geq n$ or u and v are adjacent, then the graph has a Hamilton circuit.

  Proof idea: Suppose there is any graph that had this property but no Hamilton cycle. Consider such a graph that has as the maximum number of edges without having a Hamilton cycle. Such a graph must have a Hamilton path: if not, we could add more edges without creating a cycle.

  By the pigeonhole principle, there must be vertices adjacent to the ends of the path in such a way that we can construct a circuit. [Google “Ore's Theorem” for details of the proof if you're interested.]
- **Corollary:** (Dirac's Theorem) In a graph with $n \geq 3$ vertices, if each vertex has $\deg(v) \geq n/2$, then the graph has a Hamilton circuit.

  Proof: If a graph has $\deg(v) \geq n/2$ for each vertex, then it meets the criteria for Ore's theorem, and thus has a Hamilton cycle.∎
- Note that these conditions are sufficient but not necessary: there are graphs that have Hamilton circuits but do not meet these conditions.
  - $C6$ for example (cycle with 6 vertices): each vertex has degree 2 and 2<6/2, but there is a Ham cycle.

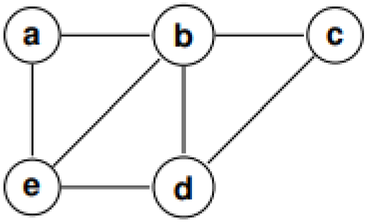


- There are no nice necessary-and-sufficient conditions known for a graph to have a Hamilton circuit.
  - Just a few more that give imperfect conditions on one side or the other.
- There is also no good algorithm known to find a Hamilton path/cycle.
  - The most obvious: check every one of the $n!$ possible permutations of the vertices to see if things are joined up that way.
  - There are known algorithms with running time $O(n22n)$ and $O(1.657n)$.

- Either way, they're exponential, so we're not going to come up with a solution for a large graph.
- There's no proof that no non-exponential algorithm exists, either.

Example :

Determine whether the given graph has an Hamilton circuit. If it does, find such a circuit. It it does not, give an argument to show why no such circuit exists.



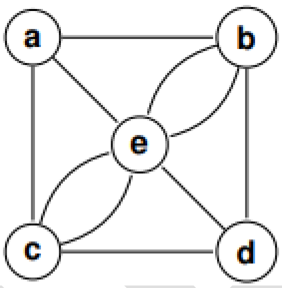
Solution:

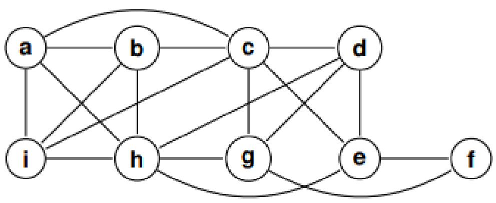
This graph has a Hamilton circuit. a, b, c, d, e, a is a circuit.

4.5 Exercise

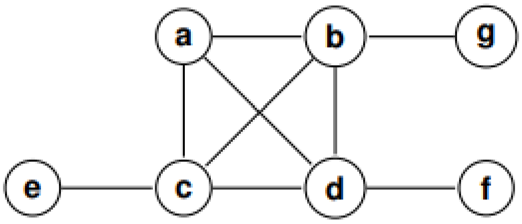
Solve the following :

1) Determine whether the given graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, determine whether the graph has an Euler path and construct such a path if one exists

a) 

b) 

2) Determine whether the given graph has an Hamilton circuit. If it does, find such a circuit. It it does not, give an argument to show why no such circuit exists.



Unit 4 - Chapter 5

Mathematical Models

Chapter Overview:

5.0 Objective

We can describe A mathematical model as a system using mathematical concepts and language. Mathematical modelling is the process of developing a mathematical model. These methods are used in the natural sciences like physics, biology, earth science, chemistry and engineering disciplines such as computer science, electrical engineering.

Mathematical models are also used in non-physical systems such as the social sciences such as economics, psychology, sociology, political science.

As well as they are also used in music,linguistics and philosophy.

A model may help to explain a system and to study the effects of different components, and to make predictions about behavior.

5.1 Mathematical Models

- In mathematical modelling, we translate our beliefs about how the world functions into the language of mathematics.

- **Advantages of using mathematical models are:**

  1. Mathematics is a very precise language. This helps us to formulate ideas and identify underlying assumptions.

  2. Mathematics is a concise language, with well-defined rules for manipulations.

3. All the results that mathematicians have proved over hundreds of years are at our disposal.

4. Computers can be used to perform numerical calculations.

- We need to make huge compromises while building any mathematical modelling.
- The very first level of compromise is to identify the most important parts of the system.These will be included in the model, the rest will be excluded.
- The second level of worthwhile compromise concerns the amount of mathematical manipulation required.

- Mathematics can prove general results, though these results depend critically on the form of equations used. Small changes in the structure of equations may require enormous changes in the mathematical methods.
- Using computers to handle the model equations may never lead to elegant results, but it is much more robust against alterations.
- Mathematical modelling can be used for a number of different reasons.
- How well any particular objective is achieved depends on both the state of knowledge about a system and how well the modelling is done.
- Examples of the range of objectives are:

  1. Developing scientific understanding - through quantitative expression of current knowledge of a system ,as well as displaying what we know, this may also show up what we do not know

  2. test the effect of changes in a system

  3. aid decision making, including

     (i) tactical decisions by managers; (ii) strategic decisions by planners.

5.2 Vehicular Stopping Distance Modelling using decision theory

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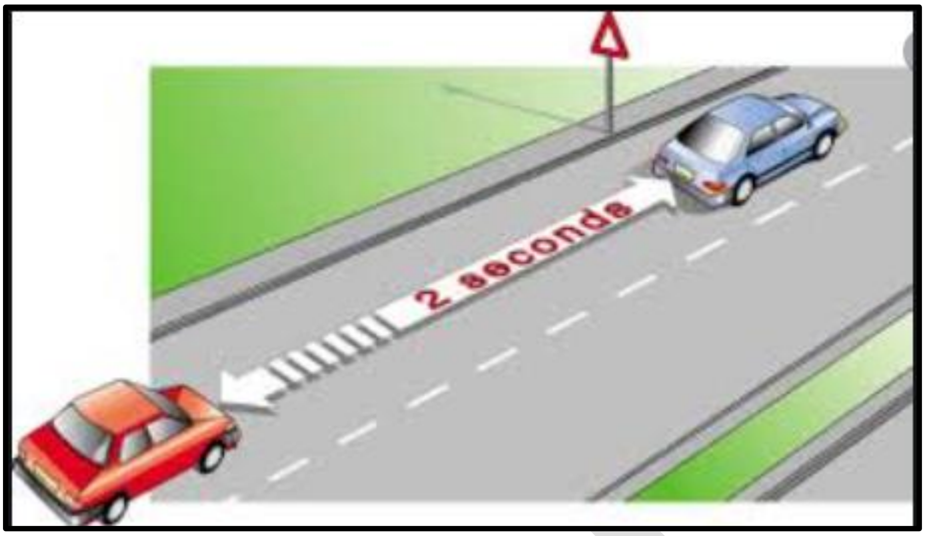
Let's discuss mathematical modelling by following an example.

Example. Vehicular Stopping Distance



Problem statement :

In driver's training, you learn a rule for how far behind other cars you are supposed to stay.

- ➔ Stay back one car length for every 10 mph of speed.
- ➔ Use the two-second rule: stay two seconds behind.

This is an easy-to-follow rule; it is a safe rule?

Steps to follow :

1. **Formulation**:

   State the question.

   Identify factors.
2. **Describe mathematically**:

   Culminates with a mathematical model.
3. **Mathematical Manipulation**

   Determine mathematical conclusions.
4. **Evaluation.**

   Translate into real-world conclusions.

   How good is the model?

5.2.1 Formulation:

6.2.1.1 State the question.

First, we need to state the question/questions clearly and precisely.

- Is the two-second rule the same as the 10 mph rule?

- Does the two-second rule mean we'll stop in time?
- Determine the total stopping distance of a car as a function of its speed.

6.2.1.2 Identify factors

Now we need to identify factors that influence our problem statement.

Stopping distance is a function of what?

v velocity

t_r reaction time

a vehicle acceleration / deceleration

5.2.2 Describe mathematically:

Now breaking down the problem Describe mathematically.

Subproblem 1:

Determine reaction distance d_r

Assume speed is constant throughout reaction distance.

Then,

reaction distance is $d_r = t_r \cdot v$.

Subproblem 2: **Total**

Determine stopping distance d_b

Assume brakes applied constantly throughout stopping, producing a constant deceleration. Brake force is $F = ma$, applied over a breaking distance d_b.

This energy absorbs the kinetic energy of the car,½ mv.

Solving

$m \cdot a \cdot db = ½\, mv^2$,

we expect $db = C \cdot v^2$.

Therefore,

The total stopping distance is

$$d_r + d_b = t_r.v + Cv^2$$

5.2.3 Mathematical Manipulation

Now we have a mathematical model.But still did we answer the question?

We need to determine the reaction distance and stopping distance.

Does data already exist?

If not, can we gather data?

Data is available from US Bureau of Public Roads.

Examine methodology of data collection.

Experimenters said t_r = 3/4 sec and calculated d_r

Perhaps we should design our own trial?

5.2.4 Evaluation.

5.2.4.1 Translate into real-world conclusions

- We can compile data in a range.
- Trials ran until had a large enough sample
- Then middle 85% of the trials given.
- We're modeling d_b as a function of v^2, so transform the x-axis.
- We can answer the following:
  - Do we try to fit to low value, avg value, or high value in range to achieve our goal "prevent accidents"?

- We conclude that the total stopping distance is
- $d_{tot} = d_r + d_b \approx 1.1v + 0.054v^2$.

- **Conclusion:**
  - Plots observed stopping distance versus model.
  - Model seems reasonable (through 70 mph).
  - Residual plot shows additional behavior unmodeled

5.2.4.2 How good is the model?

To check how good is the model we can ask the below questions and try to answer them:

1. Is the model accurate?
   Basically yes. Our model gives a slight overestimate of total stopping time through 70 mph.
2. Is the model precise?
   Yes, the model gives a definite answer.
3. Is the model descriptively realistic?
   Yes, the model was created based on the physics of stopping.
4. Is the model robust?
   Our model is not robust.

5.3 Limitations in our model

We need to answer the following question by assuming:

Is the model general?

When is it reasonable?

What are its limitations?

- ➔ Drivers going ≤ 70 mph
- ➔ Good road conditions
- ➔ Applies when driving cars, not trucks.
- ➔ Current car manufacturing; revise every five years
- ➔ In the future, perhaps there will be no accidents.

5.4 Conclusion : Vehicular Stopping Distance Model

Does the two-second rule mean that we'll stop in time?

- ➔ Recognize that a two-second rule is Easy to implement.

- ➔ The two-second rule is a linear rule,
- ➔ A quadratic rule would make more sense.
- ➔ Works up until 40 mph, then quickly invalid!

Come up with a variable rule based on speed.

- ➔ It's not reasonable to tell people to stay 2.5 seconds behind at 50 mph and 2.8 seconds behind at 58 mph!
- ➔ Determine speed ranges where
    - ◆ two seconds is enough (≤ 40 mph)
    - ◆ three seconds enough (≤ 60 mph)
    - ◆ four seconds enough (≤ 75 mph)
    - ◆ Add more if non-ideal road conditions.

5.5 Exercise

Answer the following:

1) Write a short note on a mathematical model.
2) Explain Vehicular Stopping Distance Modelling using decision theory.
3) Is Vehicular Stopping Distance Modelling is good? Justify your answer.
4) Write limitations of Vehicular Stopping Distance Model**.**
5) What is decision theory? Explain briefly.

Unit 4 - Chapter 6
Probability

Chapter Overview:

6.0 Objective

- Probability theory is a mathematical modeling of the phenomenon of chance or randomness.
- For example, if a coin is tossed in a random manner, it can land heads or tails, but we do not know which of these will occur in a single toss.
- However stable long-run behavior of random phenomena forms the basis of probability theory.
- A probabilistic mathematical model of random phenomena is defined by assigning "probabilities" to all the possible outcomes of an experiment.
- The reliability of our mathematical model for a given experiment depends upon the closeness of the assigned probabilities to the actual limiting relative frequencies.
- In this section we state the axioms, derive a few consequences, and introduce the notion of expected value.

6.1 Probability

- Using a mathematical theory of probability, we may be able to calculate the likelihood of some event.
- Many fields in computer science such as machine learning, cryptography, computational linguistics, computer vision, robotics, and of course algorithms, rely a lot on probability theory.
- In1933, Kolmogorov provided a precise axiomatic approach to probability theory which made it into a rigorous branch of mathematics with even more applications.

- Probability can be conceptualized as finding the chance of occurrence of an event. Mathematically, it is the study of random processes and their outcomes.
- The first basic assumption of probability theory is that even if the outcome of an experiment is not known in advance, the set of all possible outcomes of an experiment is known.
- This set is called the **sample space** or **probability space**.

Common Terms related to Probability theory are as follow:

- **Random Experiment:** An experiment in which all possible outcomes are known and the exact output cannot be predicted in advance is called a random experiment. Tossing a fair coin is an example of random experiment.
- **Sample Space:** When we perform an experiment, then the set S of all possible outcomes is called the sample space. If we toss a coin, the sample space S = {H, T}
- **Event:** Any subset of a sample space is called an event. After tossing a coin, getting Head on the top is an event.
- **Probability** means the chance of occurrence of a particular event. The best we can say is how likely they are to happen, using the idea of probability.

$$\text{Probability of occurrence of an event} = \frac{\text{Total number of favourable outcome}}{\text{Total number of Outcomes}}$$

Steps to find the probability:

1: Calculate all possible outcomes of the experiment.
2: Calculate the number of favorable outcomes of the experiment.
3: Apply the corresponding probability formula.

Conditional Probability:

- The conditional probability of an event B is the probability that the event will occur given an event A has already occurred. This is written as P(B|A).
- If event A and B are mutually exclusive, then the conditional probability of event B after the event A will be the probability of event B that is P(B).

$$\mathbf{P(B|A) = P(A \cap B) / P(A)}$$

Example 1:

In a country 50% of all teenagers know English and 30% of all teenagers know English and Marathi. What is the probability that a teenager knows English given that the teenager owns a Marathi?

Solution:

Let us assume A is the event of teenagers knowing only a Marathi and B is the event of teenagers knowing only a English.

So,

P(A) = 50/100
= 0.5 and

P(A ∩ B) = 30/100
= 0.3

from the given problem.

P(B|A) = P(A ∩ B) / P(A)
= 0.3/0.5
= 0.6

Hence, the probability that a teenager knows English given that the teenager knows a Marathi is 60%.

Example 2:

In a class, 50% of all students play cricket and 25% of all students play cricket and volleyball. What is the probability that a student plays volleyball given that the student plays cricket?

Solution

Let us assume A is the event of students playing only cricket and B is the event of students playing only volleyball.

So,

P(A) = 50/100
=0.5
and

P(A ∩ B) = 25/100
=0.25

from the given problem.

P(B|A) = P(A ∩ B) / P(A)
=0.25/0.5
=0.5

Hence, the probability that a student plays volleyball given that the student plays cricket is 50%.

Mutually exclusive events:

Mutually exclusive events are those where the occurrence of one indicates the non-occurrence of the other

OR

When two events cannot occur at the same time, they are considered mutually exclusive.

Note: For a mutually exclusive event, P(A and B) = 0.

Example: Consider the example of finding the probability of selecting a black card or a 6 from a deck of 52 cards.

Solution:

We need to find out P(B or 6)

Probability of selecting a black card = 26/52

Probability of selecting a 6 = 4/52

Probability of selecting both a black card and a 6 = 2/52

P(B or 6) = P(B) + P(6) – P(B and 6)

= 26/52 + 4/52 – 2/52

= 28/52

= 7/13.

Independent and Dependent Events:

Independent Event

When multiple events occur, if the outcome of one event DOES NOT affect the outcome of the other events, they are called independent events.

Say, a die is rolled twice. The outcome of the first roll doesn't affect the second outcome. These two are independent events.

Example 1: Assume a coin is tossed twice. What is the probability of getting two consecutive tails ?

Solution:

Probability of getting a tail in one toss = 1/2

The coin is tossed twice. So 1/2 * 1/2 = 1/4 is the answer.

Here's the verification of the above answer with the help of sample space.

When a coin is tossed twice, the sample space is {(H,H), (H,T), (T,H), (T,T)}.

Our desired event is (T,T) whose occurrence is only once out of four possible outcomes and hence, our answer is 1/4.

Example 2: Consider another example where a pack contains 4 blue, 2 red and 3 black pens. If a pen is drawn at random from the pack, replaced and the process repeated 2 more times, What is the probability of drawing 2 blue pens and 1 black pen?

Solution

Here, total number of pens = 9

Probability of drawing 1 blue pen = 4/9

Probability of drawing another blue pen = 4/9

Probability of drawing 1 black pen = 3/9

Probability of drawing 2 blue pens and 1 black pen = 4/9 * 4/9 * 3/9

= 48/729

= 16/243

Dependent Events

When two events occur, if the outcome of one event affects the outcome of the other, they are called dependent events.

Consider the aforementioned example of drawing a pen from a pack, with a slight difference.

Example 1: A pack contains 4 blue, 2 red and 3 black pens. If 2 pens are drawn at random from the pack, NOT replaced and then another pen is drawn. What is the probability of drawing 2 blue pens and 1 black pen?

Solution:

Probability of drawing 1 blue pen = 4/9

Probability of drawing another blue pen = 3/8

Probability of drawing 1 black pen = 3/7

Probability of drawing 2 blue pens and 1 black pen = 4/9 * 3/8 * 3/7

= 1/14

Example 2: What is the probability of drawing a king and a queen consecutively from a deck of 52 cards, without replacement.

Solution:

Probability of drawing a king = 4/52 = 1/13

After drawing one card, the number of cards are 51.

Probability of drawing a queen = 4/51.

Now, the probability of drawing a king and queen consecutively is = 1/13 * 4/51

= 4/663

6.2 Expected Value

Consider an experiment that has only two possible events. The expected value of the experiment is,

Expected value = (probability of event 1)(payoff for event 1) + (probability of event 2)(payoff for event 2)

Finding an Expected Value Involving Two Events:
The expected value of an "experiment" is the long-run average,if the experiment could be repeated many times, the expected value is the average of all the results.

Expected value E(x) - (Mean of a random variable):

- Decision or taking a specific action.
- If the expected value is positive, a gain is expected.
- If it is negative, a loss is expected.
- The expected value represents the average gain (profit) or loss if the same decision is repeated over and over again.
- In application, the expected value is typically illustrated through 'games of chance'.
- InIt is often used to predict how much is likely to be gained or lost based on making a certain d general, it can be calculated as,

$$E(x) = \sum [x \cdot P(x)]$$

OR

$$E(x) = \sum(Profit)(Probability\ of\ Profit) - \sum(Cost)(Probability\ of\ Cost)$$

6.3 Examples:

Tossing a Coin:

If a coin is tossed, there are two possible outcomes: Heads (H) or Tails (T)

Solution:

Total number of outcomes = 2

Hence, the probability of getting a Head (H) on top is ½ and

the probability of getting a Tails (T) on top is ½

Throwing a Dice:

When a dice is thrown, six possible outcomes can be on the top: 1, 2, 3, 4, 5, 6.

Solution:

The probability of any one of the numbers is 1/6

The probability of getting even numbers is 3/6=1/3

The probability of getting odd numbers is 3/6=1/3

Taking Cards From a Deck:

From a deck of 52 cards, if one card is picked, find the probability of an ace being drawn and also find the probability of a diamond being drawn.

Solution:

Total number of possible outcomes: 52

Outcomes of being an ace: 4

Probability of being an ace = 4/52 =1/13

Probability of being a diamond = 13/52 =1/4

Life Insurance Policies:

Example1:

An insurance company sells a one-year term life insurance policy to a 75-year old woman. The woman pays a premium of $750. If she dies within 1 year, the insurance company will pay $25,000 to her beneficiary. According to the 2014 Social Security Actuarial Life Table, the probability that a 75-year old woman will be alive in 1 year is 0.9746. Find the expected value of the insurance company's profit on the policy. Interpret the result.

Solution:

$$E(x)\ of\ Profit = (\$750)(0.9746) - (\$25{,}000 - \$750)(1 - 0.9746) = \$115$$

The amount required to purchase the policy	The probability that the woman lives for at least 1 year	The amount the insurance company "loses" if the woman dies	The probability of the woman dying within 1 year

Interpretation - *This indicates that if the insurance company sells many policies like this one, the company can expect on average to earn a $115 profit per policy. (Gain)*

Example2:

You take out a fire insurance policy on your home. The annual premium is $300. In case of fire, the insurance company will pay you $200,000. The probability of a house fire in your area is 0.0002.

a. What is the expected value?

b. What is the insurance company's expected value?

c. Suppose the insurance company sells 100,000 of these policies. What can the company expect to earn?

Solution:

a. Expected value = (0.0002)(199,700) + (0.9998)(−300) = − $260.00

Fire No Fire

The expected value over many years is −$260 per year.

b. The expected value for the insurance company is the same, except the perspective is switched.

Instead of − $260 per year, it is +$260 per year.

Of this, the company must pay a large percent for salaries and overhead.

c. The insurance company can expect to gross $30,000,000 in premiums on 100,000 such policies.

With a probability of 0.0002 for fire, the company can expect to pay on about 20 fi res.

This leaves a gross profit of $26,000,000.

Calculate Roulette Probabilities:

For roulette purposes, a factorial shows in how many different ways, different items (or numbers) can be arranged. Without repetitions of the same item or number. To give you an idea how huge this number can become, for 37 numbers, like in European roulette:

$37! = 1.3763753 \times 1043$

THE PROBABILITY EQUATION

Here is the principal mathematical formula for calculating the chance of any roulette outcome or event.

First we must define the parameters:

P(e) is the probability of an event E.

n is the number of trials (spins)

x is the number of times our bet wins

P(b) is the probability of our bet B winning in one spin

The probability **P(e)** of the event **E = [Bet b appearing x times in n spins] =**

$$\frac{(n)!}{((x)! \times ((n-x))!)} \times P(b)^{x} \times (1 - P(b))^{(n-x)}$$

SIMPLE EXAMPLE:

Let's say that we want to calculate the probability of two Blacks in three spins. Or to put it differently "how often will we see **exactly** two Black numbers in three spins". Note that this equation calculates the exact probabilities of a specific event. Not the probabilities of 2 *or more* Blacks, but *exactly* 2 Blacks.

Solution:

The parameters are:

n = 3 (total spins)

x = 2 (Black numbers/winning spins)

P(b) = 0.5 (the probability of Black in each spin – we ignore the zero for simplicity)

P(e) = (n!/(x!(n-x)!)) P(b)x (1-P(b))n-x
P(e) = (3!/(2!(3-2)!)) 0.52 (1-0.5)3-2
P(e) = (3!/(2!1!)) 0.52 0.51
P(e) = (3×~~2×1~~/~~2×1~~×1) 0.25× 0.5
P(e) = (3/1) 0.25× 0.5
P(e) = 3 × 0.125 = **0,375**

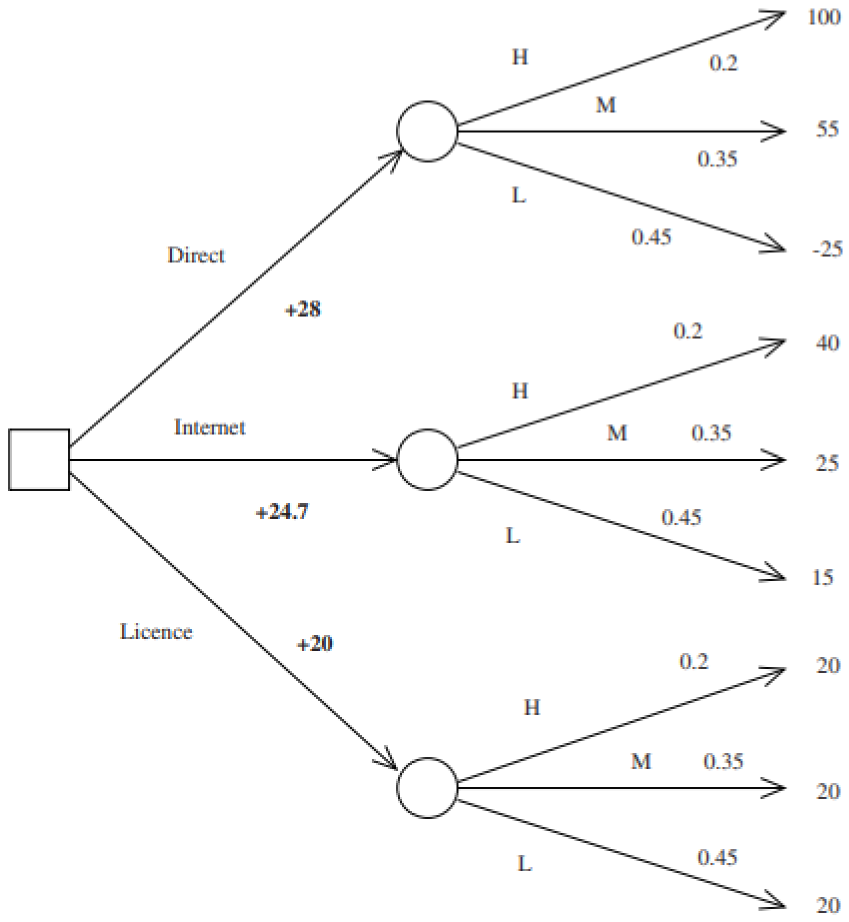
Therefore, the probability in 3 spins to have exactly 2 Black numbers is **0,375** or **37,5%** or a little more than 1/3.

All these numbers are just different expressions of the same thing ,the expectation of the event happening.

6.4 Decision Trees

- Decision tree is a hierarchical tree structure that is used to classify classes based on a series of questions (or rules) about the attributes of the class.
- The attributes of the classes can be any type of variables from binary, nominal, ordinal, and quantitative values, while the classes must be qualitative (categorical or binary, or ordinal).
- In short, given a data of attributes together with its classes, a decision tree produces a sequence of rules (or series of questions) that can be used to recognize the class.
- When we include a decision in a tree diagram we use a rectangular node, called a decision node to represent the decision.
- The diagram is then called a decision tree.
- There are no probabilities at a decision node but we evaluate the expected monetary values of the options.
- In a decision tree the first node is always a decision node.
- If there is another decision node then we evaluate the options there and choose the best one, and the expected value of this option becomes the expected value of the branch leading to the decision node.

- The decision tree example would look like this:



Use of decision tree:
Decision tree can be used to predict a pattern or to classify the class of a data. Suppose we have new unseen records of a person from the same location where the data sample was taken. The following data are called test data (in contrast to training data) because we would like to examine the classes of these data.

A general algorithm for a decision tree can be described as follows:

1. Pick the best attribute/feature. The best attribute is one which best splits or separates the data.
2. Ask the relevant question.
3. Follow the answer path.
4. Go to step 1 until you arrive at the answer.

The best split is one which separates two different labels into two sets.

Example1:
Your company is considering developing one of two cell phones. Your development and market research teams provide you with the following projections.
Cell phone A:

Cost of development: \$2,500,000

Projected sales: 50% chance of net sales of \$5,000,000

30% chance of net sales of \$3,000,000

20% chance of net sales of \$1,500,000

Cell phone B:

Cost of development: \$1,500,000

Projected sales: 30% chance of net sales of \$4,000,000

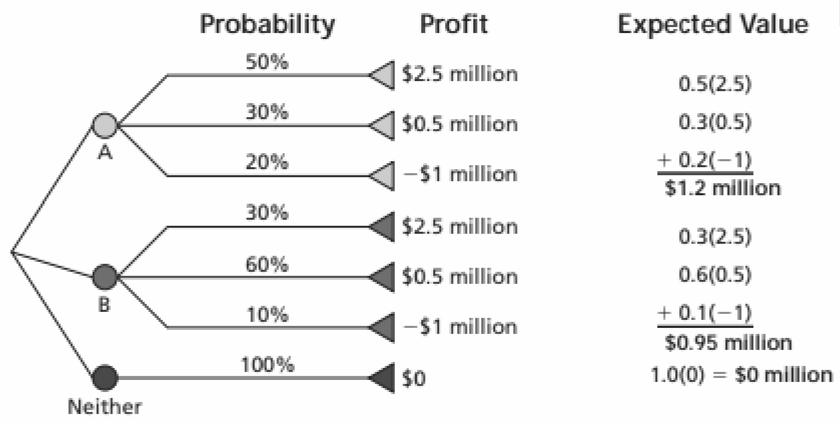
60% chance of net sales of \$2,000,000

10% chance of net sales of \$500,000

Which model should your company develop? Explain.

Solution:

A decision tree can help organize your thinking.



Although cell phone A has twice the risk of losing \$1 million, it has the greater expected value. So, using expected value as a decision guideline, your company should develop cell phone A.

Example 2:

Teams A and B are to play each other repeatedly until one wins two games in a row or a total of three games. One way in which this tournament can be played is for A to win the first game, B to win the second, and A to win the third and fourth games. Denote this by writing A–B–A–A.

a. How many ways can the tournament be played?

b. Assuming that all the ways of playing the tournament are equally likely, what is the probability that five games are needed to determine the tournament winner?

Solution:

a. The possible ways for the tournament to be played are represented by the distinct paths from “root” (the start) to “leaf” (a terminal point) in the tree shown sideways in the given figure.

The label on each branching point indicates the winner of the game.

The notations in parentheses indicate the winner of the tournament.

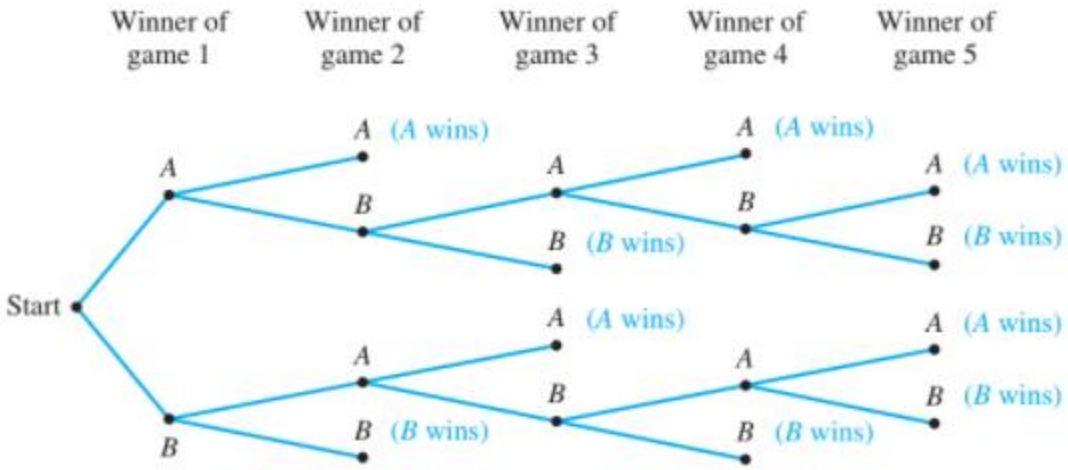


Figure 9.2.1 The Outcomes of a Tournament

The fact that there are ten paths from the root of the tree to its leaves shows that there are ten possible ways for the tournament to be played. They are (moving from the top down): A–A, A–B–A–A, A–B–A–B–A, A–B–A–B–B, A–B–B, B–A–A,B–A–B–A–A, B–A–B–A–B, B–A–B–B, and B–B. In five cases A wins, and in the other five B wins. The least number of games that must be played to determine a winner is two, and the most that will need to be played is five.

b. Since all the possible ways of playing the tournament listed in part (a) are assumed to be equally likely, and the listing shows that five games are needed in four different cases (A–B–A–B–A, A–B–A–B–B, B–A–B–A–B, and B–A–B–A–A),
the probability that five games are needed is 4/10 = 2/5 = 40%.

7.5 Classification problems using Baye's theorem

Theorem: If A and B are two mutually exclusive events, where P(A) is the probability of A and P(B) is the probability of B, P(A | B) is the probability of A given that B is true.

$$P(A \mid B) = \frac{P(B \mid A)\,P(A)}{\sum_{i=1}^{n} P(B \mid Ai)P(Ai)}$$

Application of Bayes' Theorem:

- ➔ In situations where all the events of sample space are mutually exclusive events.
- ➔ In situations where either P(Ai ∩ B) for each Ai or P(Ai) and P(B|Ai) for each Ai is known.

Problem
Consider a three box container.

The first box container contains 2 red flowers and 3 blue flowers;
the second box container has 3 red flowers and 2 blue flowers; and
the third box container has 4 red flowers and 1 blue flower.
There is equal probability of each box container to be selected. If one flower is drawn at random, what is the probability that it is a red flower?

Solution:
Let Ai be the event that i^{th} box container is selected.
Here, i = 1,2,3.
Since probability for choosing a box container is equal, P(Ai) = 1/3

Let B be the event that a red flower is drawn.

The probability that a red flower is chosen among the five flower of the first box container:
P(B|A1) = 2/5

The probability that a red flower is chosen among the five flower of the second box container:
P(B|A2) = 3/5

The probability that a red flower is chosen among the five flower of the third box container:
P(B|A3) = 4/5

According to Bayes' Theorem,

$$P(B) = P(A1) . P(B|A1) + P(A2) . P(B|A2) + P(A3) . P(B|A3)$$
$$= 1/3 \cdot 2/5 + 1/3 \cdot 3/5 + 1/3 \cdot 4/5$$
$$= 3/5$$

6.6 Exercise:

Solve the following:

1. Picoplex Technologies have developed a new manufacturing process which they believe will revolutionise the smartphone industry. They are, however, uncertain how they should go about exploiting this advance. Initial indications of the likely success of marketing the process are 55%, 30%, 15% for "high success", "medium success" and "probable failure", respectively. The company has three options; they can go ahead and develop the technology themselves, licence it or sell the rights to it. The financial outcomes (in £ millions) for each option are given in the table below.

	"high success"	"medium success"	"failure"
Develop	80	40	–100
Licence	40	30	0
Sell	25	25	25

(a) Draw a decision tree to represent the company's problem.

(b) Calculate the Expected Monetary Value for all possible decisions the company may take and hence determine the optimal decision for the company.

2. Which of the following should your company develop? Explain.

   Cell phone C:

   Cost of development: $2,000,000

   Projected sales: 40% chance of net sales of $5,000,000
   40% chance of net sales of $3,000,000
   20% chance of net sales of $1,500,000

   Cell phone D:

   Cost of development: $1,500,000

   Projected sales: 15% chance of net sales of $4,000,000
   75% chance of net sales of $2,000,000
   10% chance of net sales of $500,000

3. What is the probability of throwing a 5 with a die?
4. If one has two dice, what is the probability of throwing a 5 with the first die and a 6 with the other die?
5. We toss a coin, either heads or tails might turn up, but not heads and tails at the same time.

Unit 5: Modelling Using Difference Equations
Chapter 7

Unit Structure

7.0Overview

In this chapter, we will introduce the idea of recurrence relations (difference equation). We will study some of the famous recurrence relations like Fibonacci Sequence, Tower of Hanoi, Lines in plane. We will study homogenous and non-homogeneous difference equations with their solutions. Then we will study Divide and Conquer algorithms.

7.1Introduction

In this chapter, we are going to study the differenceequations and their applications with solution. To understand the difference equation, we consider some basic terms. The recurrence relations are also called as Difference Equations.

- **Sequence:**

  The sequence is a function from set of natural numbers to real numbers defined as $f\colon \mathbb{N} \to \mathbb{R}$ with $f(n) = a_n$, where $n \in \mathbb{N}$ and $a_n \in \mathbb{R}$ and is denoted by $< a_n >$

  E.g., Let$f\colon \mathbb{N} \to \mathbb{R}$defined as $f(n) = 2n$be a sequence $< a_n >$is defined as $a_n = 2n$, $n = \mathbb{N}$ and $a_n \in \mathbb{R}$. We get the sequence as $< a_n > = \{2, 4, 6, 8, \ldots.\}$, where $a_1 = 2, a_2 = 4, a_3 = 6$ ... and so on.

7.2.1 Recurrence Relation:

Let $< a_n >$ be the sequence of real numbers and if we are able to express the terms of $< a_n >$ in terms of previous terms of $< a_n >$ itself then such relation od terms is called Recurrence Relation.

E.g.

1. We consider the previous sequence $< a_n >$ where $a_n = 2n$. Now we have the sequence $< a_n >$ as $\{2, 4, 6, 8, \ldots\}$

Here if we consider $a_1 = 2, a_2 = 4$then, we can write $a_2 = a_1 + 2$

Also $a_3 = 6$,here we have $a_3 = a_2 + 2$ and so on.

In general, we can write $a_{n+1} = a_n + 2$

This is the recurrence relation for the given sequence of numbers.

2. Now we consider an integer sequence $\{1, 2, 3, 6, 11, 20, 37, \ldots\}$ with

$a_1 = 1, a_2 = 2, a_3 = 3$ and the general recurrence relation defined as

$a_n = a_{n-1} + a_{n-2} + \ a_{n-3}$ for all $n \geq 4$.

Here we can see that

$$a_4 = a_3 + a_2 + \ a_1 = 3 + 2 + 1 = 6$$
$$a_5 = a_4 + a_3 + \ a_2 = 6 + 3 + 2 = 11$$

and $a_6 = a_5 + a_4 + \ a_3 = 11 + 6 + 3 = 20$ and so on.

7.2.2 Fibonacci Series:

Consider the sequence with $F_1 = 1$ and $F_2 = 1$ which has a recursive relation defined as $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

The sequence generated by this recursive relation is called as Fibonacci Sequence and is given as

$$F_3 = F_2 + F_1 = 1 + 1 = 2$$
$$F_4 = F_3 + F_2 = 2 + 1 = 3$$
$$F_5 = F_4 + F_3 = 3 + 2 = 5$$

$F_6 = F_5 + F_4 = 5 + 3 = 8$ and so on.

We get $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$

7.2.3 Lucas Numbers:

Consider the sequence generated by the recurrence relation

$L_n = L_{n-1} + L_{n-2}$ where $n \geq 3$ and $L_1 = 2, L_2 = 1, \ldots,$ for $n \in \mathbb{N}$.

We have the sequence as

$$L_3 = L_2 + L_1 = 1 + 2 = 3$$
$$L_4 = L_3 + L_2 = 3 + 1 = 4$$

$L_5 = L_4 + L_3 = 4 + 3 = 7$ and so on.

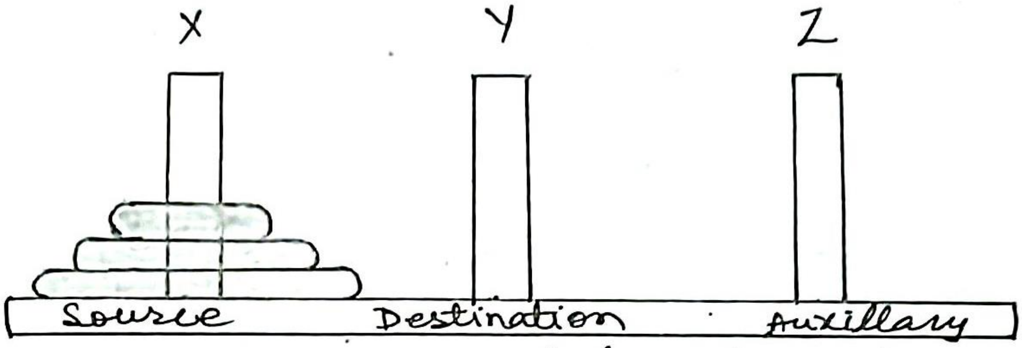
We get the sequence as $\{2, 1, 3, 4, 7, 11, 18, 29, \ldots\}$.

These numbers are called as Lucas Numbers. This sequence of Lucas numbers can also be generated from the recurrence relation of Fibonacci Sequence.

7.3 Tower of Hanoi:

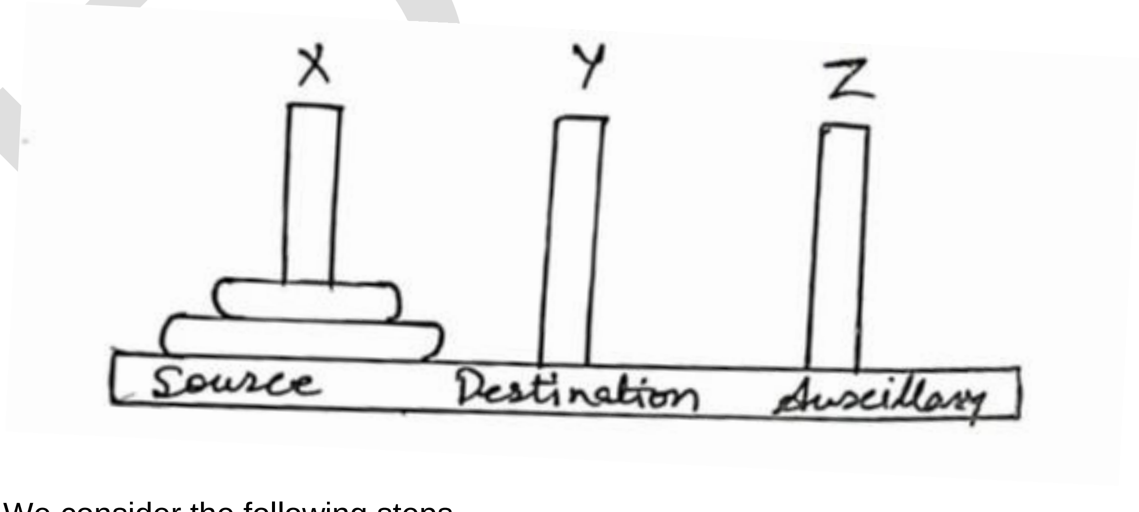
Tower of Hanoi is a mathematical puzzle also relates to the ancient Tower of Brahma. According to the literature, when the world was created, there was a diamond tower (X) with 64 golden discs. The discs were in decreasing size and stacked on tower X in decreasing size from bottom to top. There are two more towers Y and Z. From the time of creation of world, the priests have been attempting to move the discs from tower X (source) to tower Y (destination) using tower Z (auxiliary). These discs should be moved with the following rules.

1. Only one disc can be moved at a time among the towers.
2. Only the disc on the top can be moved.
3. At any time the larger disc cannot be moved on the smaller disc.



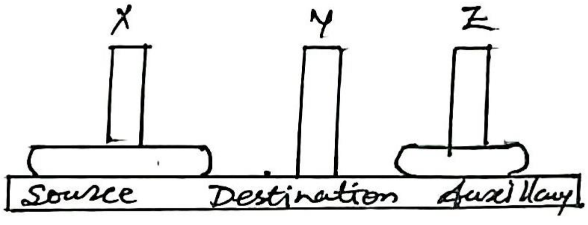
According to the literature, the world will come to an end when all the discs will be moved to the tower Y. If we want to move all discs od tower X (source) to tower Y (destination) then how many minimum steps (moves) do we need? We can answer this question with an algorithm. We need to construct the algorithm.

We first try to solve this problem with less discs. Let we have 2 discs only. Now we consider the 3 towers X(source), Y (destination) and Z(auxiliary).

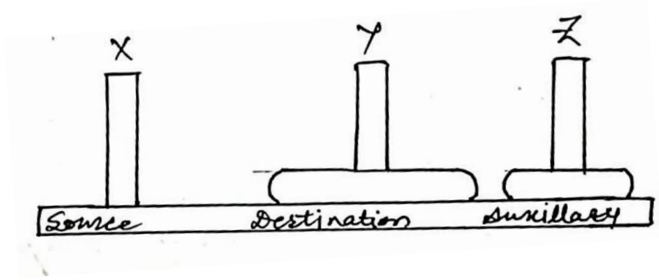


We consider the following steps.

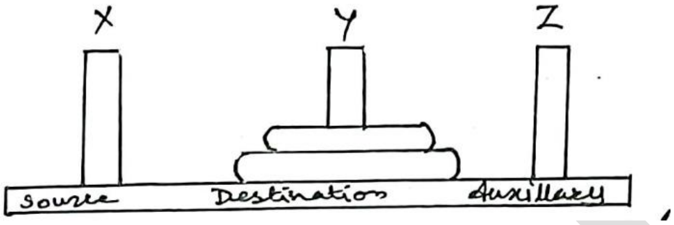
a. Move the smaller disc to tower Z from tower X.



b. Move the larger disc to tower Y from tower X



c. And finally move the smaller disc from tower Z to tower Y



Now we have got the idea, how to solve this puzzle. Let's try to get the algorithm to solve Tower of Hanoi for more than 2 discs.

We consider the stack of discs into 2 parts. The largest disc (n^{th} disc) is the one part and other (n-1) disc are the other part.

We have to move all the discs from tower X(source) to the tower Y (destination).

We can use the above steps to move the discs from tower X to tower Y. Now we apply the above steps as

a. Move (n-1) discs (second part) from tower X to tower Z.
b. Move n^{th} (largest) disc from tower X to tower Y
c. Move all (n-1) discs from tower Z to tower Y.

This algorithm we can apply recursively to all n discs and move all discs to destination tower Y.

If we use proper recursion then we are able to solve the puzzle with n discs in minimum 2^n-1 steps.

As we know for 2 discs, we have solved this puzzle in 2^2-1=3 steps. Hence we can use the induction to solve the problem with n disks.

Here we have the recurrence relation as

$$H_n = 2H_{n-1} + 1 \text{ for } n \geq 2 \text{ and } H_1 = 1.$$

Example 1: Find a recurrence relation and initial conditions for the number of bit strings of the length n that do not have two consecutive zeroes(0's). How many such bit strings are there of length 5?

Solution: Let a_n denote the number of bit strings of length n that do not have consecutive 0's. We have to find the recursive relation for the sequence $< a_n >$.

Now by the sum rule, the number of bit strings of length n that do not have two consecutive zeroes equal to the number of such bit strings ending with '0' (zero) and the numbers of such bit strings ending with'1'.

Consider that $n \geq 3$ so that the bit string has at least 3 bits.

Consider the initial conditions $a_1 = 2$. Since if we consider the but strings of length 1 then we have either 0 or 1 as both the strings do not have consecutive zeroes.

Now we consider $n = 2$ i.e., the bit strings with length 2. We have the bit strings of length 2 as 01, 10 and 11 i.e., we get $a_2 = 3$.

Now we try to generalise the result as below.

The bit strings of length n ending with 1 that do not have consecutive 0 are precisely the bit strings of length $(n-1)$ with no two consecutive zeroes with 1 added at the end and we have such a_{n-1} strings.

Now the bit strings of length n ending with '0' that do not have consecutive 0's must have 1 at their $(n-1)^{th}$ place (bit), otherwise these will end with 00. Hence, we have the number of such bit strings of length n ending with 0 that have no consecutive '0' are the bit strings of length $(n-2)$ and we add 10 at the last place. There are such a_{n-2} strings. Hence, we have total number of bit strings of length n with no two consecutive zeroes is a_n

$$\therefore a_n = a_{n-1} + a_{n-2} \text{ for n≥3} \qquad \text{-------}(1)$$

Now we want to find the nos. of bit strings with length 5.

We have $a_1 = 2, \quad a_2 = 3,$

$\therefore a_3 = a_2 + a_1$ from equation (1)

$\therefore a_3 = 3 + 2 = 5$

Also $\therefore a_4 = a_3 + a_2 = 5 + 3 = 8$

and $\therefore a_5 = a_4 + a_3 = 8 + 5 = 13$

$\therefore$ we have such 13 bit-strings of length 5.

Ex.2 Find the first 6 terms of the sequence of recurrence relation

$$a_n = 6a_{n-1} \text{ where } a_0 = 2.$$

(Hint: Here we have to find the terms up to a_5.)

Solution: We have the recurrence relation as

$$a_n = 6a_{n-1} \text{ and } a_0 = 2.$$

Consider the first 6 terms of the sequence as,

For n=1, $\therefore a_1 = 6a_0 = 6(2) = 12$

For n=2, $\therefore a_2 = 6a_1 = 6(12) = 72$

For n=3, $\therefore a_3 = 6a_2 = 6(72) = 432$

For n=4, $\therefore a_4 = 6a_3 = 6(432) = 2592$

For n=5, $\therefore a_5 = 6a_4 = 6(2592) = 15552.$

Ex.3 Find the first 5 terms of the sequence of recurrence relation

$$a_n = a_{n-1} + 3a_{n-2} \text{ with } a_1 = 1, a_2 = 2.$$

Solution: We have the recurrence relation as

$$a_n = a_{n-1} + 3a_{n-2} \quad \text{for n} \geq 2$$

Consider the first 6 terms of the sequence as,

For n=2 $\therefore a_2 = a_1 + 3a_0 = 2 + 3(1) = 5$

For n=3 $\therefore a_3 = a_2 + 3a_1 = 5 + 3(2) = 11$

For n=4 $\therefore a_4 = a_3 + 3a_2 = 11 + 3(5) = 26$

For n=5 $\therefore a_5 = a_4 + 3a_3 = 26 + 3(11) = 59.$

Ex.4 Find the a_5 term of the sequence of recurrence relation

$$a_n = a_{n-1} + a_{n-3} \text{ with } a_0 = 1, a_1 = 2 \text{ and } a_2 = 0.$$

Solution: We have the recurrence relation as

$$a_n = a_{n-1} + a_{n-3}$$

with $a_0 = 1, a_1 = 2$ and $a_2 = 0$.

Now consider

$$a_3 = a_2 + a_0 = 0 + 1 = 1$$
$$a_4 = a_3 + a_1 = 1 + 2 = 3$$
$$a_5 = a_4 + a_2 = 3 + 0 = 3$$
$$\therefore\ a_5 = 3$$

Ex.5 Find first 6 terms of the recurrence relation $a_n = -2a_{n-1}$ with $a_0 = -1$

Solution: We have the recurrence relation as $a_n = -2a_{n-1}$ with $n \geq 1$ and $a_0 = -1$

Now consider

$$a_1 = -2a_0 = -2(-1) = 2$$
$$a_2 = -2a_1 = -2(2) = -4$$
$$a_3 = -2a_2 = -2(-4) = 8$$
$$a_4 = -2a_3 = -2(8) = -16$$
$$a_5 = -2a_4 = -2(-16) = 32$$
$$\therefore a_5 = 32$$

Ex.6 Find the terms up to a_5 for the recursive relation $a_n = a_{n-1} - a_{n-2}$ with the initial conditions $a_0 = 2$ and $a_1 = -1$

Solution.: We have the recurrence relation as $a_n = a_{n-1} - a_{n-2}$ with the initial conditions $a_0 = 2$ and $a_1 = -1$

Now we have

$$a_2 = a_1 - a_0 = -1 - 2 = -3$$
$$\therefore a_3 = a_2 - a_1 = -3 - (-1) = -3 + 1 = -2$$
$$\therefore a_4 = a_3 - a_2 = -2 - (-3) = -2 + 3 = 1$$
$$\therefore a_5 = a_4 - a_3 = 1 - (-2) = 1 + 2 = 3$$
$$\therefore a_5 = 3$$

Ex.7 Find first 5 terms of recursive relation $a_n = 2a_{n-1} + 3$ with $a_0 = 3$.

Solution: We have the recursive relation as $a_n = 2a_{n-1} + 3$ for $n \geq 1$ with $a_0 = 3$

Now we consider

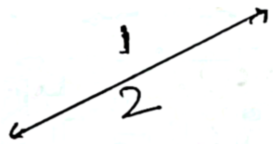
$$a_1 = 2a_0 + 3 = 2(3) + 3 = 9$$
$$a_2 = 2a_1 + 3 = 2(9) + 3 = 21$$
$$a_3 = 2a_2 + 3 = 2(21) + 3 = 45$$
$$a_4 = 2a_3 + 3 = 2(45) + 3 = 93$$
$$\therefore a_4 = 93$$

7.4 Lines in a Plane:

Let a_n denote the number of regions defined by n lines in a plane i.e., if we draw n lines in a plane then the plane will be divided into a_n regions. Now we consider the smaller cases of n.

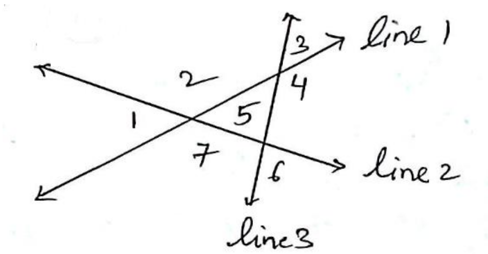
Let $n = 0$, so we have the whole plane as one region $\therefore\ a_0 = 1$.

Now $n = 1$, if we draw one line in a plane then we see that the plane is divided into 2 regions $\therefore\ a_1 = 2$



Now we consider $n = 2$. Now if we draw 2 lines in a plane then we see that the plane is divided into 4 regions. Hence here we get $a_2 = 4$.

Here we need to understand that every time we draw a line it does not double the number of regions. When we draw a third line, we see that $a_3 = 7$



No matter how we draw the line, it can split at the 3 old regions.

Now we try to generalise the relation as recurrence relation. The n^{th} line (for $n > 0$), increases the number of regions by k if and only if it splits k of the old regions. It splits k old regions if and only if it intersects the previous lines in $k - 1$ different places. Two lines can intersect at the most at one point. Therefore, the new line can intersect the $n - 1$ lines in at the most $n - 1$ different points, and we must have $k \leq n$.

Hence, we can have a recurrence relation as $a_n \leq a_{n-1} + n$ for $n > 0$

Here we have inequality, now we will try to convert it to equality using induction. Now we draw the n^{th} line in such a way that it's not parallel to any of the others hence it intersects them all, and that it doesn't go through any of the existing intersection points (hence, it intersects all of the lines).

The recurrence is given as $a_0 = 1$, $a_n = a_{n-1} + n$ for $n > 0$

We have the values $a_0 = 1, a_1 = 2$ and $a_2 = 4$ satisfy the above relation.

Now we simplify the above relation $a_n = a_{n-1} + n$ and $a_{n-1} = a_{n-2} + (n-1)$

$$\therefore a_n = [a_{n-2} + (n-1)] + n$$

$$= a_{n-2} + (n-1) + n \qquad \because a_{n-2} = a_{n-3} + (n-2)$$

$$= [a_{n-3} + (n-2)] + (n-1) + n$$

$$= a_{n-3} + (n-2) + (n-1) + n \qquad \because a_{n-3} = a_{n-4} + (n-3)$$

$$\therefore a_n = a_{n-4} + (n-3) + (n-2) + (n-1) + n$$

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$$a_n = a_0 + 1 + 2 + 3 + \cdots + (n-3) + (n-2) + (n-1) + n$$

Now we know that

$$1 + 2 + 3 + \cdots + (n-2) + (n-1) + n = \frac{n(n+1)}{2}$$

i.e., sum of first n-natural numbers.
Hence, we have
$a_n = a_0 + \frac{n(n+1)}{2}$ for all $n \geq 0$

$$\because a_0 = 1$$

We get

$$a_n = 1 + \frac{n(n+1)}{2}$$

$\therefore a_n = \frac{1}{2}(n)(n+1) + 1$ for all $n \geq 0$
The recurrence relations are also called as difference equations.

7.5 Homogenous Linear Equations with Constant Coefficients:

Definition: Let $k \in \mathbb{N}$ and $C_0(\neq 0), C_1, C_2, \ldots C_k(\neq 0)$ be the constants. If a_n, for $n \geq 0$, is a discrete function then
$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n)$, $n \geq k$
is called Linear Recursion Relation with constant coefficient of order k.
When $f(n) = 0 \ \forall \ n \geq 0$ then the relation is called Homogenous.
When $f(n) \neq 0 \ \forall \ n \geq 0$ then the relation is called Non-homogenous.
E.g.:

a. Let $2a_n - 3a_{n-1} = 2^n$ is a first order linear recursive relation with constant coefficients.
b. The recursive relation $3a_n - 5a_{n-1} + 2a_{n-2} = n^2 + 5$ is a second order linear recurrence relation with constant coefficient.
c. The recursive relation $a_n = 2a_{n-1}$, $n \geq 1$ is the homogenous recurrence relation as we have $a_n = 2a_{n-1} = 0$. The degree (order) of this recurrence relation is one.
d. The recurrence relation of Fibonacci numbers $F_n = F_{n-1} + F_{n-2}$ is also a homogenous recurrence relation we often studied as they are in many modelling problems.

7.5.1 Solving Linear Homogenous Recurrence Relations with constant coefficients.

Let
$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k}$ $\ldots (i)$
be the homogenous recurrence relation which is linear.
We look for the solution of above equation of the form $a_n = m^n$ for some constant m.
Replacing the terms by $a_n = m^n$ is above equation (i), we get

$$m^n = C_1 m^{n-1} + C_2 m^{n-2} + \cdots + C_k m^{n-k}$$

Dividing above equation by m^{n-k}, we have

$$m^k = C_1 m^{k-1} + C_2 m^{k-2} + \cdots + C_{k-1} m + C_k$$

$\therefore m^k - C_1 m^{k-1} - C_2 m^{k-2} - \cdots - C_{k-1} m - C_k = 0 \qquad \ldots(ii)$

The equation (ii) is called as Characteristic Equation of homogenous linear recurrence relation (i).

The solutions of characteristic equation are called as Characteristic roots of recurrence relation.

The sequence $< a_n >$ with $a_n = m^n$ is a solution if and only if $'m'$ is solution of equation (ii).

7.5.2 Theorem: First Order Linear Homogenous Recurrence Relation.

Let the first order linear homogenous recurrence relation with constant coefficient be defined as

$a_{n+1} = d a_n$ for $n \geq 0$ and d is constant with the initial condition $a_0 = k$

then the general solution is given by

$\therefore\ a_n = k d^n$ for $n \geq 0$

This is the unique solution for the given initial condition.

Ex.1: Solve the recurrence relation $a_{n+1} = 3a_n$ for $n \geq 0$ and $a_0 = 5$.

Solution: Let $a_{n+1} = 3a_n$ for $n \geq 0$ be the given linear homogenous recurrence relation with $a_0 = 5$.

Since the above relation is of order 1, i.e., first order, hence, by the theorem, the general solution is given as $a_n = k d^n$ for $n \geq 0$

Here $k = a_0 = 5$ and $d = 3$.

We get

$a_n = 5.3^n \quad$ for $n \geq 0$

Ex.2: A bank pays 6% (annual) interest on savings, compounding the interest monthly. If Reeta deposits Rs. 1000 on first day of May, how much will this deposit be worth a year later?

Solution: Let 6% be the annual rate of interest given by bank on deposits. So monthly rate is $\frac{6}{12}\% = 0.5\% = 0.005$.

Let P_n be the Reeta's deposit at the n months $0 \leq n \leq 12$. Then we have the recurrence relation as

$P_{n+1} = P_n + 0.005 P_n \qquad$ [$\because P_n$ is deposit and $0.005 P_n$ is interest]

$\therefore P_{n+1} = 1.005 P_n \qquad \ldots(i)$

$\therefore P_{n+1} = 1.005 P_n \qquad$ for $0 \leq n \leq 11$

And $P_0 = 1000$

Here relation (i) is linear homogenous recurrence relation of order 1. Hence by theorem, the unique solution of (i) is given as

$$P_n = k d^n$$

Here $k = P_0 = 1000$ and $d = 1.005$

We get

$$P_n = P_0(1.005)^n$$

$\therefore P_n = 1000(1.005)^n$, $n \geq 0$ is the solution of recurrence relation (i).

Now at the end of the year Reeta will get,

For $n = 12$,

$$P_{12} = 1000(1.005)^{12}$$
$$P_{12} = Rs. 1061.68$$

Therefore, the value of Reeta's deposit at the end of the year will be Rs. 1061.68

Ex.3: Solve the following recurrence relation $a_{n+1} = 3a_n$ for $n \geq 0$ and $a_0 = 7$.

Solution: Let $a_{n+1} = 3a_n$ for $n \geq 0$ be the linear homogenous recurrence relation with $a_0 = 7$.

Hence by the theorem, the solution is given as

$a_n = kd^n$ for $n \geq 0$

Here $d = 3$ and $k = 7$

We get

$a_n = 7.3^n$ for $n \geq 0$

Ex.4: Find the unique solution to the recurrence relation $a_{n+1} - 1.5a_n = 0$ for $n \geq 0$ with $a_0 = 2$.

Solution: Let $a_{n+1} - 1.5a_n = 0$ for $n \geq 0$ be the linear homogenous recurrence relation with $a_0 = 2$.

Hence by the theorem, the unique solution is given as

$a_n = kd^n$ for $n \geq 0$

Here $d = 1.5$ and $k = 2$

We get

$a_n = 2(1.5)^n$ for $n \geq 0$

7.5.3 Linear homogenous recurrence relation of degree two:

Now we will solve linear homogenous recurrence relation of degree (order) 2.

Consider the second order recurrence relation as

$$a_{n+2} = c_1 a_{n+1} + c_2 a_n$$

To get the characteristic equation, we put $a_{n+2} = m^{n+2}, a_{n+1} = m^{n+1}$ and $a_n = m^n$ in above equation, we get

$$m^{n+2} = c_1 m^{n+1} + c_2 m^n$$

Cancelling m^n on both sides we get,

$$m^2 = c_1 m + c_2$$

$\therefore$The characteristic equation of 2-degree homogenous linear recurrence relation is given as

$$m^2 - C_1 m - C_2 = 0$$

This is quadratic equation in m. Hence it has 2 roots. Depending on the nature of roots we have 3 cases as

a. Roots are real and distinct

b. Roots are real and repeated
c. Roots are complex numbers

7.5.4 Theorem: If the roots are real and distinct

Let m_1 and m_2 be the roots of characteristic equation $m^2 - C_1 m - C_2 = 0$ then the solution of linear homogenous recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2}$ is given as

$a_n = \alpha m_1^n + \beta m_2^n$ where α and β are constants.

Ex.1: Solve the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.

Solution: Let linear homogenous recurrence relation be $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.

We have the characteristic equation as

$$m^2 = m + 2$$

Note: Replace lowest order term by m^0, and the next term by m and then m^2; i.e., $'a_n'$ by m^2, $'a_{n-1}'$ by m for characteristic equation.

$$\therefore m^2 - m - 2 = 0$$

$$(m-2)(m+1) = 0$$

$\therefore m = 2, -1$ since the roots are real and distinct.

The general solution is given as $a_n = \alpha 2^n + \beta(-1)^n$ where α and β are the constant to be determined.

We have $a_0 = 2 = \alpha + \beta$ for $n = 0$ and $a_1 = 7 = 2\alpha - \beta$ for $n = 1$

Solving the above linear equations, we get $\alpha = 3$ and $\beta = -1$

Hence, we have the solution as $a_n = 3.2^n - (-1)^n$

Ex.2: Find the solution for Fibonacci numbers $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$

Solution: Consider the linear homogenous recurrence relation $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$ and $F_1 = 1$.

We have the characteristic equation as

$m^2 - m - 1 = 0$ [Replace $'F_n'$ by m^2, $'F_{n-1}'$ by m and $'F_{n-2}'$ by $m^0 = 1$.]

Solving above quadratic equation, we get $m = \frac{1\pm\sqrt{5}}{2}$

Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$

Since the roots of the characteristic equation are real and distinct, the general solution is given as $F_n = C_1\alpha^n + C_2\beta^n$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ and C_1 and C_2 be the constants to be determined.

We have $F_0 = 0$ and $F_1 = 1$

For $n = 0, F_0 = 0 = C_1 + C_2$ and

For $n = 1, F_1 = 1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)$

Solving above linear equations, we get $C_1 = \frac{1}{\sqrt{5}}$ and $C_2 = -\frac{1}{\sqrt{5}}$.

$\therefore$ The general solution for Fibonacci sequence is given as

$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$ for $n \geq 0$

7.5.5 Theorem: If the roots are real and repeated:

Let m be the characteristic root which is repeated twice and real then the solution of linear homogenous recurrence relation $a_n = C_1 a_{n-1} + C_2 a_{n-2}$ is given as
$a_n = (\alpha + \beta n)m^n$, where α and β are constants

Ex.1: Solve the recurrence relation $a_n = 6a_{n-1} + 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$
Solution: Consider the linear homogenous recurrence relation $a_n = 6a_{n-1} + 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$
We have the characteristic equation as $m^2 - 6m - 9 = 0$
Solving above quadratic equation, we get $(m-3)^2 = 0$
$\therefore m = 3, 3$
Since the roots of characteristic equation are real and repeated, the general solution of recurrence relation is given as $a_n = (\alpha + \beta n)(3)^n$ for $n \geq 0$ where α and β are the constant to be determined.
Let $n = 0$, we get$a_0 = 1 = \alpha, \therefore \alpha = 1$
And $n = 1$, we have $a_1 = 6 = (\alpha + \beta)3 = 3\alpha + 3\beta$
Solving above equations we get $\alpha = 1$ and $\beta = 1$.
∴ The general solution of the homogenous linear recurrence relation is given as
$a_n = (n+1)3^n$ for $n \geq 0$

Ex.2: Solve the recurrence relation $a_n = 4a_{n-1} + 4a_{n-2}$ with $a_0 = 1$ and $a_1 = 3$.
Solution: Let $a_n = 4a_{n-1} + 4a_{n-2}$ be the homogenous recurrence relation with $a_0 = 1$ and $a_1 = 3$
We have the characteristic equation as

$$m^2 - 4m + 4 = 0$$
$$\therefore \ (m-2)^2 = 0$$
$$\therefore m = 2, 2$$

Since the roots of characteristic equation are real and repeated, the general solution of recurrence relation is given as
$a_n = (\alpha + \beta n)(2)^n$ for $n \geq 0$
Where α and β are the constants to be determined.
For $n = 0$, we get$a_0 = 1 = \alpha, \therefore \alpha = 1$
And $n = 1$, we have $a_1 = 3 = (\alpha + \beta)(2) = 2\alpha + 2\beta$
Solving above linear equations, we have $\alpha = 1$ and $\beta = \frac{1}{2}$
∴ the general solution of homogenous linear recurrence relation is

$$a_n = \left(1 + \frac{1}{2}n\right)(2)^n = 2^n + \frac{n}{2}2^n = 2^n + n2^{n-1}$$

$\therefore a_n = 2^n + n2^{n-1}$ for $n \geq 0$

7.5.6 Theorem: If the roots are complex:

Let $x \pm iy$ be the roots of characteristic equation of a linear homogenous recurrence relation then the general solution is given as

$$a_n = r^n[\alpha \cos(n\theta) + \beta \sin(n\theta)]$$

Where α and β are the constants and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and $r = \sqrt{x^2 + y^2}$

Ex.1: Solve the recurrence relation $a_n = 2a_{n-1} - 2a_{n-2}$ with $a_0 = 1$ and $a_1 = 2$.

Solution: Let $a_n = 2a_{n-1} - 2a_{n-2}$ be the linear homogenous recurrence relation with $a_0 = 1$ and $a_1 = 2$

We have the characteristic equation as

$$m^2 - 2m + 2 = 0$$
$$m^2 - 2m + 1 + 1 = 0$$
$$(m-1)^2 + 1 = 0$$
$$(m-1)^2 = -1$$
$$m - 1 = \pm\sqrt{-1} = \pm i$$
$$m = 1 \pm i$$

The roots of the characteristic equation are complex roots. Consider the polar form of complex numbers as

$$1 + i = r(\cos\theta + i\sin\theta)$$

Where $r = \sqrt{x^2 + y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$\therefore r = \sqrt{1^2 + 1^2} = \sqrt{2}$ and $\theta = \tan^{-1}\left(\frac{1}{1}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

$\therefore$ we have general solution as

$a_n = r^n[\alpha \cos(n\theta) + \beta \sin(n\theta)]$ for $n \geq 0$

Here $r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

$\therefore$ we get

$a_n = (\sqrt{2})^n \left[\alpha \cos\left(\frac{n\pi}{4}\right) + \beta \sin\left(\frac{n\pi}{4}\right)\right]$ for $n \geq 0$

$a_n = 2^{\frac{n}{2}} \left[\alpha \cos(\frac{n\pi}{4}) + \beta \sin(\frac{n\pi}{4})\right]$ for $n \geq 0$

To find the values of α and β, we consider the given values of $a_0 = 1$ and $a_1 = 2$

For $n = 0$, $a_0 = 1 = \alpha\cos(0) + \beta\sin(0) = \alpha \quad \because \alpha = 1$

For $n = 1$, $a_1 = 2 = \sqrt{2}\left[1.\cos\left(\frac{\pi}{4}\right) + \beta \sin\left(\frac{\pi}{4}\right)\right] = \sqrt{2}\left[\frac{1}{\sqrt{2}} + \beta.\frac{1}{\sqrt{2}}\right] = 1 + \beta$

$$\therefore \beta = 1$$

We get $\alpha = 1$ and $\beta = 1$, hence the general solution is given as

$a_n = 2^{\frac{n}{2}} \left[\cos\left(\frac{n\pi}{4}\right) + \sin\left(\frac{n\pi}{4}\right)\right]$for $n \geq 0$

Ex.2: Solve the recurrence relation $a_n = a_{n-1} - 4a_{n-2}$ with $a_0 = 1$ and $a_1 = 1$.

Solution: Let $a_n = a_{n-1} - 4a_{n-2}$ be the linear homogenous recurrence relation with $a_0 = 1$ and $a_1 = 1$.

Consider the characteristic equation

$$m^2 - m + 4 = 0$$

Solving above quadratic equation we get,

$$m = \frac{1 \pm i\sqrt{15}}{2}$$

We have

$$x + iy = \frac{1}{2} + i\frac{\sqrt{15}}{2}$$

$$\therefore r = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{15}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{15}{4}} = \sqrt{\frac{16}{4}} = \sqrt{4} = 2$$

$$\therefore r = 2$$

And $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{15}/2}{1/2}\right) = \tan^{-1}\left(\sqrt{15}\right)$

$$\therefore \theta = \tan^{-1}\left(\sqrt{15}\right)$$

Since the roots are complex roots, we have the general solution as
$a_n = r^n[\alpha \cos(n\theta) + \beta \sin(n\theta)]$ for $n \geq 0$
Here $r = 2$
$\therefore\ a_n = 2^n[\alpha \cos(n\theta) + \beta \sin(n\theta)]$ for $n \geq 0$
Where $\theta = \tan^{-1}\left(\sqrt{15}\right)$ and α and β are constants.
To find the value of α and β, we consider the initial values of a_0 and a_1
For $n = 0, a_0 = 0 = \alpha \cos(0) + \beta \sin(0) = \alpha$

$$\therefore \alpha = 0$$

For $n = 1$, $a_1 = 1 = 2\alpha \cos\theta + 2\beta \sin\theta$
Since $\alpha = 0$

$$\therefore 1 = 2\beta \sin\theta$$

We know that $\sin\theta = \frac{2\tan(\theta/2)}{1+tan^2(\theta/2)}$

Using above formula and solving we get $\beta = \frac{2}{\sqrt{15}}$

Hence the general solution is given as

$a_n = 2^n.\frac{2}{\sqrt{15}}\sin(n\theta)$ for $n \geq 0$

$\therefore a_n = \frac{2^{n+1}}{\sqrt{15}}\sin n\theta$ for $n \geq 0$

And $\theta = \tan^{-1}\left(\sqrt{15}\right)$

7.5.7 Theorem: General Solution:

Let $C_1, C_2, \ldots, C_k$ be the real numbers and we have the characteristic equation.
$m^k - C_1 m^{k-1} - C_2 m^{k-2} - \cdots - C_k = 0$ with k distinct roots $m_1, m_2, \ldots, m_k$ then the general solution of corresponding homogenous linear recurrence relation is given as
$a_n = \alpha_1 m_1^n + \alpha_2 m_2^n + \cdots + \alpha_k m_k^n$ for $n \geq 0$
Where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_k$ are constants.

Ex.1: Solve the recurrence relation $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$ for $n \geq 0$ with $a_0 = 0, a_1 = 1$ and $a_2 = 2$

Solution: Let $2a_{n+3} = a_{n+2} + 2a_{n+1} - a_n$ be the recurrence relation with $a_0 = 0, a_1 = 1$ and $a_2 = 2$

This is the linear homogenous recurrence relation. Consider the characteristic equation of above recurrence relation.

$$\therefore 2m^3 - m^2 - 2m + 1 = 0$$
$$\therefore 2m^3 - 2m - m^2 + 1 = 0$$
$$\therefore 2m(m^2 - 1) - (m^2 - 1) = 0$$
$$\therefore (m^2 - 1)(2m - 1) = 0$$
$$\therefore (m - 1)(m + 1)(2m - 1) = 0$$

$\therefore m = 1, -1$ and $\frac{1}{2}$

$\therefore$ The roots of characteristic equation are $-1, \frac{1}{2}, 1$. All roots are distinct, hence the general solution is given as

$a_n = \alpha_1(-1)^n + \alpha_2(1/2)^n + \alpha_3(1)^n$ for $n \geq 0$

Where α_1, α_2 and α_3 are the constants to be determined.

To determine the values of α_1, α_2 and α_3 we consider the given initial values $a_0 = 0, a_1 = 1$ and $a_2 = 2$

For $n = 0, \therefore\ a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 0$

For $n = 1, \therefore\ a_1 = 1 = -\alpha_1 + \frac{\alpha_2}{2} + \alpha_3$

$\therefore\ -2\alpha_1 + \alpha_2 + 2\alpha_3 = 2$

And

For $n = 2, \therefore\ a_2 = 2 = \alpha_1 + \frac{\alpha_2}{4} + \alpha_3$

$\therefore\ 4\alpha_1 + \alpha_2 + 4\alpha_3 = 8$

Solving above linear equations, we get

$\alpha_1 = \frac{1}{6}, \alpha_2 = -\frac{8}{3}$ and $\alpha_3 = \frac{5}{2}$

Hence the solution is given as

$a_n = \frac{1}{6}(-1)^n - \frac{8}{3}\left(\frac{1}{2}\right)^n + \frac{5}{2}$ for $n \geq 0$

Ex.2: Solve the recurrence relation $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ for $n \geq 3$ with $a_0 = 1, a_1 = -2$ and $a_2 = -1$

Solution: Let $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$ for $n \geq 3$ be the linear homogenous recurrence relation with $a_0 = 1, a_1 = -2$ and $a_2 = -1$

Consider the characteristic equation of above recurrence relation, we have

$$m^3 + 3m^2 + 3m + 1 = 0$$
$$(m + 1)^3 = 0$$
$$\therefore m = -1, -1, -1$$

i.e., the root -1 is repeated 3 times. Hence the general solution is given as

$a_n = \alpha_1 m^n + \alpha_2.n.m^n + \alpha_3.n^2 m^n$ for $n \geq 0$

Where α_1, α_2 and α_3 are the constants to be determined and $m = -1$.

We have

$a_n = \alpha_1(-1)^n + \alpha_2.n.(-1)^n + \alpha_3.n^2(-1)^n$ for $n \geq 0$

We have the initial values as $a_0 = 1, a_1 = -2$ and $a_2 = -1$

For $n = 0,\ a_0 = 1 = \alpha_1,\ \therefore \alpha_1 = 1$
For $n = 1, a_1 = -\alpha_1 - \alpha_2 - \alpha_3 = -2,\ \therefore \alpha_1 + \alpha_2 + \alpha_3 = 2$
For $n = 2,\ a_2 = -1 = \alpha_1 + 2\alpha_2 + 4\alpha_3$
Solving above linear equations, we get $\alpha_1 = 1,\ \alpha_2 = 3$ and $\alpha_3 = -2$
Hence the general solution is given as
$a_n = (1 + 3n - 2n^2)(-1)^n$ for $n \geq 0$

Ex.3: Find the solution of $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ with $a_0 = 2, a_1 = 5$ and $a_2 = 15$

Solution: Let $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ be the linear homogenous recurrence relation with $a_0 = 2, a_1 = 5$ and $a_2 = 15$

Consider the characteristic equation

$$m^3 - 6m^2 + 11m - 6 = 0$$
$$(m-1)(m-2)(m-3) = 0$$
$$\therefore m = 1, 2, 3$$

The roots of characteristic equation are 1, 2 and 3.
Hence the general solution of recurrence relation is given as
$a_n = \alpha_1(1)^n + \alpha_2(2)^n + \alpha_3(3)^n$ for $n \geq 0$
Where α_1, α_2 and α_3 are the constants to be determined.
To find the values of α_1, α_2 and α_3, we consider the initial conditions a_0, a_1 and a_2.
For $n = 0, \therefore\ a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$
For $n = 1, \therefore\ a_1 = 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3$
And
For $n = 2, \therefore\ a_2 = 15 = \alpha_1 + 4\alpha_2 + 9\alpha_3$
Solving the above linear equations, we get $\alpha_1 = 1,\ \alpha_2 = -1$ and $\alpha_3 = 2$
Hence the solution is given as
$a_n = 1 - 2^n + 2.3^n$ for $n \geq 0$

7.6 Linear Non-homogenous Recurrence Relations with Constant Coefficients:

The linear recurrence relation of the form
$a_n = C_1a_{n-1} + C_2a_{n-2} + \cdots + C_ka_{n-k} + f(n)$ ... (i)
Where $C_1, C_2, \ldots, C_k$ are constants and $f(n)$ is function of only n, is called Linear Non-homogenous recurrence relation of order k.
The recurrence relation
$a_n = C_1a_{n-1} + C_2a_{n-2} + \cdots + C_ka_{n-k}$ ... (ii)
is called as Associated homogenous recurrence relation.

7.6.1 Homogenous Solution:

The solution of Associated Homogenous linear recurrence relation is called Homogenous solution and is denoted by $a_n^{(h)}$.
i.e., The solution $a_n^{(h)}$ is a solution of linear homogenous recurrence relation
$a_n = C_1a_{n-1} + C_2a_{n-2} + \cdots + C_ka_{n-k}$ of order(degree) k.

7.6.2 Particular Solution:

The solution of recurrence relation (i) due to the term $f(n)$, is called Particular Solution and is denoted as $a_n^{(p)}$. The particular solution depends on $f(n)$.

7.6.3 Total Solution:

The total solution of non-homogenous linear recurrence relation is given as the sum of homogenous solution (the solution of associated homogenous linear recurrence relation) and the particular solution (the solution due to the function $f(n)$ of n only)

i.e., Total Solution $=a_n^{(h)} + a_n^{(p)}$

7.6.4 Theorem: General solution of Non-HomogenousLinear recurrence relation:

If $\left\{a_n^{(p)}\right\}$ is a particular solution of the non-homogenous linear recurrence relation with constant coefficients

$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} + f(n)$ ... (i)

Then every solution is of the form $\left\{a_n^{(h)} + a_n^{(p)}\right\}$, where $a_n^{(h)}$ is the solution of the associated homogenous linear recurrence relation

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k}$$

Ex.1: Solve the following recurrence relation $a_n = 3a_{n-1} + 2n$ with $a_1 = 3$

Solution: Let

$a_n = 3a_{n-1} + 2n$ ... (1)

be the non-homogenous linear recurrence relation with $a_1 = 3$.

Consider the associated homogenous linear recurrence relation

$a_n = 3a_{n-1}$ ... (2) for $n \geq 2$ and $a_1 = 3$

We have the characteristic equation as

$$m - 3 = 0$$

$$\therefore m = 3$$

$\therefore$ The solution of associated homogenous linear recurrence relation is given as

$a_n^{(h)} = \alpha.3^n$ ... (3)

Where α is constant to be determined.

Now we consider the particular solution.

We have $f(n) = 2n$ is of degree 1 in $'n'$.

Hence, we consider the particular solution as polynomial of degree 1 in $'n'$.

i.e., we have $a_n = Cn + d$

We get $a_{n-1} = C(n-1) + d$

Using the values of a_n and a_{n-1} in equation (1), we get

$$Cn + d = 3[C(n-1) + d] + 2n$$

$$Cn + d = 3C(n-1) + 3d + 2n$$

$$Cn + d = 3Cn - 3C + 3d + 2n$$

$$Cn + d - 3Cn + 3C - 3d - 2n = 0$$
$$[C - 3C - 2]n + [d + 3C - 3d] = 0$$
$$[-2C - 2]n + (3C - 2d) = 0$$

Since $n \neq 0$, hence we get $-2C - 2 = 0$ and $3C - 2d = 0$

Solving above equations for C and d, we get

$C = -1$ and $d = -\frac{3}{2}$

Hence the particular solution is given as

$$a_n^{(p)} = -1.n - \frac{3}{2}$$

$a_n^{(p)} = -n - \frac{3}{2}$ ... (4)

From equation (3) and (4), the total solution is given as,

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$\therefore\ a_n = \alpha.3^n - n - \frac{3}{2}$ for $n \geq 1$

Where α is a constant.

To find the value of α, we consider the initial condition $a_1 = 3$.

At $n = 1$, we have,

$$a_1 = 3 = \alpha.3 - 1 - \frac{3}{2}$$
$$3 = 3\alpha - \frac{5}{2}$$
$$3\alpha = 3 + \frac{5}{2} = \frac{11}{2}$$
$$\therefore \alpha = \frac{11}{6}$$

Hence the solution of recurrence relation (1) is

$a_n = \frac{11}{6}.3^n - n - \frac{3}{2}$ for $n \geq 1$

7.6.5 Theorem: Types of Particular Solutions:

Suppose $\{a_n\}$ satisfies the linear non-homogenous recurrence relation

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} + f(n)$$

Where $C_1, C_2, \ldots, C_k$ are real constants and

$f(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) S^n$, where $b_1, b_2, \ldots, b_t$ and S are real numbers.

i) When S is not a root of the characteristic equation of associated linear homogenous recurrence relation, there is a particular solution of the form
$$(P_t n^t + P_{t-1} n^{t-1} + \cdots + P_1 n + P_0) S^n$$

ii) When S is a root of the characteristic equation and its multiplicity is m then the particular solution is of the form $n^m (P_t n^t + P_{t-1} n^{t-1} + \cdots + P_1 n + P_0) S^n$.

We consider the following table to understand the type of particular solution.

$f(n)$	$a_n^{(P)}$ *Particular Solution*

$C, a\ constant$	$A, a\ constant$
n	$A_1 n + A_0$
n^2	$A_2 n^2 + A_1 n + A_0$
n^t	$A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0$
$r^n, r \in \mathbb{R}$	Ar^n
$\sin(n\theta)$ / $\cos(n\theta)$	$A\cos(n\theta) + B\sin(n\theta)$
$n^t r^n$	$r^n(A_t n^t + A_{t-1} n^{t-1} + \cdots + A_1 n + A_0)$

From the above table it will be easy to write particular solution.

Ex.1: Solve the recurrence relation $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$.

Solution: Let

$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$ ... (1)

be the given non-homogenous linear recurrence relation.

Consider the associated homogenous linear recurrence relation

$a_n + 5a_{n-1} + 6a_{n-2} = 0$ ... (2)

We have the characteristic equation of above equation as

$$m^2 + 5m + 6 = 0$$

$$\therefore (m+2)(m+3) + 0$$

$\therefore m = -2$ and $m = -3$

The roots of the characteristic equation of the associated homogenous linear recurrence relation are -2 and -3 hence the homogenous solution is given as

$a_n^{(h)} = \alpha_1(-2)^n + \alpha_2(-3)^n$ ... (3)

Where α_1 and α_2 are constants.

Now we consider particular solution.

We have $f(n) = 3n^2 - 2n + 1$

Then the particular solution is

$a_n^{(p)} = A_2 n^2 + A_1 n + A_0$ ... (4)

From equation (1), we have

$$(A_2 n^2 + A_1 n + A_0) + 5[A_2(n-1)^2 + A_1(n-1) + A_0] + 6[A_2(n-2)^2 + A_1(n-2) + A_0] = 3n^2 - 2n + 1$$

Satisfying we get

$$12A_2 n^2 - (34A_2 - 12A_1)n + (29A_2 - 17A_1 + 12A_0) = 3n^2 - 2n + 1$$

Comparing the coefficient of n^2, n and the constant terms on both sides, we get

$$12A_2 = 3$$

$$34A_2 - 12A_1 = 2$$

$$29A_2 - 17A_1 + 12A_0 = 1$$

Solving above equations, we get,

$A_2 = \frac{1}{4}, A_1 = \frac{13}{24}$ and $A_0 = \frac{71}{288}$

From equation (4), we get

$a_n^{(p)} = \frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}$ ... (5)

From equation (3) and (5), the total solution is given as

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$\therefore a_n = \alpha_1(-2)^n + \alpha_2(-3)^n + \frac{1}{4}n^2 + \frac{13}{24}n + \frac{71}{288}, n \geq 0$

Ex.2: Solve the recurrence relation $a_n - 5a_{n-1} + 6a_{n-2} = 1$

Solution: Let

$a_n - 5a_{n-1} + 6a_{n-2} = 1$ ... (1)

be the non-homogenous linear recurrence relation.

Consider the associated homogenous recurrence relation

$a_n - 5a_{n-1} + 6a_{n-2} = 0$ ... (2)

We have the characteristic equation as

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$\therefore m = 2, 3$$

The roots of characteristic equation are 2 and 3. Hence the homogenous solution is given as

$a_n^{(h)} = \alpha_1 2^n + \alpha_2 3^n$ ... (3)

Where α_1 and α_2 are the constants. Now we consider the particular solution.

We have $f(n) = 1 = constant$

From table, we have

$$a_n^{(p)} = A = constant$$

From equation (1), we get

$$A - 5A + 6A = 1$$

$$\therefore 2A = 1$$

$$\therefore A = \frac{1}{2}$$

i.e., $a_n^{(p)} = \frac{1}{2}$ ... (4)

From equation (3) and (4) the total solution is given as

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$a_n = \alpha_1 2^n + \alpha_2 3^n + \frac{1}{2}$ for $n \geq 0$

7.7 Divide and Conquer Recurrence Relations:

We encounter many problems of recursive relation such that we are unable to solve them as they are. But if we decide the problem into smaller recursive relations and we are able to solve such smaller recursive relations then such method of dividing the larger (bigger) problem into smaller recursive relation and solving smaller recurrence relation is called as ‘Divide and Conquer Algorithms’.

E.g.: If we consider the binary search algorithm then it divides the list of entire elements in two smaller lists of small size than the original list and it successively goes on. Then we continue this process till we get the desired results. This is a ‘Divide and Conquer Algorithm’.

Suppose that the recursive algorithm divides a problem of size $'n'$ into $'a'$

subproblems of size $\frac{n}{b}$ and a total of $g(n)$ extra operations are required in conquer step of algorithm. If $f(n)$ gives the number of operations to solve the problem of size $'n'$ then we have

$$f(n) = a.f\left(\frac{n}{b}\right) + g(n)$$

This is called 'Divide and Conquer Recurrence Relation'.

Ex. 1: Find divide and conquer recurrence relation for binary search.

Solution: Let $f(n)$ denote the number of operations required to solve the problem of size $'n'$. If we consider the binary search then we divide the list into two parts. Hence it becomes of size $\frac{n}{2}$ for the next division i.e., the problem of size $'n'$ reduces now to the problem of size $\frac{n}{2}$. Now 2 comparisons are required to move further. Now we have the problem of size $'n'$ can be solved with $f(n)$ operations which is same as solving the problem of $\frac{n}{2}$ size with $f\left(\frac{n}{2}\right)$operations and 2 comparisons.

Hence the 'Divide and Conquer recurrence relation' is given as

$$f(n) = f\left(\frac{n}{2}\right) + 2$$

Where n is even.

Ex. 2: Find divide and conquer recurrence relation for the problem to find maxima and minima in a sequence.

Solution: Let $f(n)$ be the total number of operations required to find the maxima and minima of the sequence with $'n'$ elements.

We know that the problem of size n can be reduced to two problems of size $\frac{n}{2}$. Hence, we to use two comparisons, one for maxima and one for minima.

Let $f\left(\frac{n}{2}\right)$ be the number of operations required to solve the problem of size $\frac{n}{2}$ where n is even.

Hence the 'Divide and Conquer recurrence relation' is given as

$$f(n) = 2.f\left(\frac{n}{2}\right) + 2$$

Where n is even.

7.7.1 Fast Multiplication of Integers:

The divide and conquer technique is more efficient than other techniques for the fast multiplication of integers. The fast multiplication algorithm divides $2n$-bit integers into two parts each of length n-bits.

This reduces the original multiplication of two $2n$-bit integers to 3 multiplications of n-bit integers, plus shifts and additions.

Let $'a'$ and $'b'$ be two integers with binary representation of length $2n$. (We may add initial bits of zero if required to make them of length $2n$.)

Hence, we have the binary representation of $'a'$ and $'b'$ as

$a = (a_{2n-1}a_{2n-2}a_{2n-3} \dots a_1a_0)_2$ and

$$b = (b_{2n-1}b_{2n-2}b_{2n-3} \dots b_1 b_0)_2$$

Now we divide these binary representations into two parts of length n each. We get $a = 2^n A_1 + A_0$ and $b = 2^n B_1 + B_0$
where,

$$A_1 = (a_{2n-1}a_{2n-2} \dots a_{n+1}a_n)_2$$

$$A_0 = (a_{2n-1}a_{2n-2} \dots a_1 a_0)_2$$

$B_1 = (b_{2n-1}b_{2n-2} \dots b_{n+1}b_n)_2$ and

$$B_0 = (b_{2n-1}b_{2n-2} \dots b_1 b_0)_2$$

Hence, the fast multiplication of integers ab is given by

$$ab = (2^{2n} + 2^n)A_1B_1 + 2^n(A_1 - A_0)(B_0 - B_1) + (2^n + 1)A_0B_0 \quad \dots (1)$$

Suppose $f(n)$ denotes the number of operations required for the multiplication of n-bit integers.

From above multiplication (1), we have to multiply $2n$-bit integers which can be done using n-bit integers. Hence, we have 3 n-bit multiplications with additions, subtractions and shifts. These additions, subtractions and shifts can be at most n i.e., not more than n.

Hence the divide and conquer recurrence relation for fast multiplication of integers is given as

$f(2n) = 3f(n) + Cn$ where C is constant.

7.7.2 Fast Matrix Multiplication:

Let $f(n)$ denote the number of operations required to multiply 2 $n \times n$ matrices, where n is even. Now we convert this problem of $n \times n$ matrices to $\left(\frac{n}{2}\right) \times \left(\frac{n}{2}\right)$ matrices. Now this problem of multiplying 2 $n \times n$ matrices reduce to 7 multiplications of $\left(\frac{n}{2}\right) \times \left(\frac{n}{2}\right)$ matrices with 15 additions of $\left(\frac{n}{2}\right) \times \left(\frac{n}{2}\right)$ matrices.

Hence, the divide and conquer recurrence relation is given as

$f(n) = 7f\left(\frac{n}{2}\right) + 15\left(\frac{n}{2}\right)\left(\frac{n}{2}\right)$ where n is even.

$\therefore f(n) = 7f\left(\frac{n}{2}\right) + \frac{15n^2}{4}$ where n is even.

7.8 Let us Sum Up

In this chapter, we have seen the difference equations (recurrence relations). We have solved first order and second order difference equations. We have seen the applications and different real-life problems. We have also seen the algorithms like Divide and Conquer Difference Equations and their applications.

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8.10Unit End Exercises

Q. 1. Find thea_5 term of following recurrence relations (Difference Equations)

1. $a_{n+2} = a_{n+1} - 2a_n$ **with** $a_0 = 1$ **and** $a_1 = 1$. **(Ans:** $a_5 = 17$**)**
2. $a_{n+2} = a_{n+1} + 3a_n$ **with** $a_0 = 1$ **and** $a_1 = 1$. **(Ans:** $a_5 = 53$**)**
3. $a_{n+1} = a_n + a_{n-1}$ **with** $a_0 = 1$ **and** $a_1 = 1$. **(Ans:** $a_5 = 17$**)**

Q. 2. Solve the following difference equations

1. $a_{n+1} = 3a_n$, $n \geq 0$**with** $a_0 = 7$.

   (Ans: $a_n = 7(3^n)\ for\ n \geq 0$.**)**
2. $a_n = \frac{a_{n-1}}{2}\ for\ n \geq 2$ **with** $a_1 = 4$.

   (Ans: $a_n = 4\left(\frac{1}{2}\right)^{n-1}\ for\ n \geq 1$.**)**
3. $a_n = 5a_{n-1} + 6a_{n-2}$**for** $n \geq 2$ **with** $a_0 = 1$, $a_1 = 3$.

   $(Ans.\ a_n = \left(\frac{3}{7}\right)(-1)^n + \left(\frac{4}{7}\right)(6)^n\ for\ n \geq 0)$
4. $2a_{n+2} - 11a_{n+1} + 5a_n = 0$ **for** $n \geq 0$ **with** $a_0 = 2$, $a_1 = -8$.

   $(Ans.\ a_n = 4\left(\frac{1}{2}\right)^n - 2(5)^n\ for\ n \geq 0)$
5. $a_n - 6a_{n-1} + 9a_{n-2} = 0$ **for** $n \geq 2$ **with** $a_0 = 5$, $a_1 = 12$.

   $(Ans.\ a_n = (5-n)(3)^n\ for\ n \geq 0)$
6. $a_n + 2a_{n-1} + 2a_{n-2} = 0$ **for** $n \geq 2$ **with** $a_0 = 1$, $a_1 = 3$.

   $(Ans.\ a_n = \left(\sqrt{2}\right)^n \left[cos\left(\frac{3n\pi}{4}\right) + 4sin\left(\frac{3n\pi}{4}\right)\right]\ for\ n \geq 0)$
7. $a_n + 4a_{n-1} - 21a_{n-2} = 5(4^n)$**for** $n \geq 2$ **with** $a_0 = 1$, $a_1 = 3$.

   $(Ans.\ a_n = \left(\frac{-71}{10}\right)(3)^n + \left(\frac{91}{110}\right)(-7)^n + \left(\frac{80}{11}\right)(4)^n for\ n \geq 0)$
8. $a_n = 3a_{n-1} + 2n\ \ for\ \ n \geq 1$ **with** $a_0 = 1$. **(Ans:** $\frac{11}{6}(3)^n - n - \frac{3}{2}\ for\ n \geq 0$**)**
9. $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$ **for** $n \geq 2$. **(Ans:** $a_n = c_1 3^n + c_2 2^n + \left(\frac{49}{20}\right)7^n\ for\ n \geq 0$**)**

UNIT 6 - Chapter 8

CHARACTERISTICS OF COMPLEX BUSINESS PROBLEMS

Unit Structure

8.0 OBJECTIVES

After going through this unit, you will be able to:

- Describe complexity of business problem
- State the characteristics of any complex business problem statement
- Illustrate the number of possible solutions for a given problem
- Understand the concept of stable solution in time-changing environment
- Classify hard-constraints and soft-constraints
- Discuss the dominated and non-dominated solutions
- Model an MOP
- Illustrate the method for approaching and resolving a real-world problem

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8.1 INTRODUCTION

The public sector in our contemporary network society faces complex societal challenges. Attempts to deal with these challenges require the involvement and collaboration of various parties: governments, businesses, knowledge institutions, societal groups and citizens.

The following examples of cases tracking Governance and Management in the Public Sector have in commonality that they involve difficult issues. These require in-depth knowledge of the nature of the issues and possible solutions.

1) **Public infrastructural works**: Complex decision-making processes which are involved in realizing, operating and maintaining public infrastructural works. One can think of railways, roads, airports, water projects, waste incinerators, power plants, and wind turbine parks. Such infrastructural works can confront governments with a wide variety of stakeholders (private firms, citizens groups, other public stakeholders, environmental interest groups etc.). The realization of public infrastructure requires integral planning in complex networks of stakeholders, which may result in lengthy processes of negotiation and consultation.

2) **Seamless transportation:**The demand for seamless transportation requires integral planning of infrastructure investments and innovative ways of coordinating the operation of transportation services.

3) **Energy transition**: In order to deal effectively with climate change, energy transition is necessary. The development of electric cars is part of this transition and it demands a radical change in the organization of transport by car. In order to realize this innovation process, the various parties involved need to collaborate with the government, the car industry and energy producers. Meanwhile, stakeholders face resistance from the existing automobile industry while users have to be convinced of the benefits of the electric cars. These challenges require close collaboration between stakeholders and support from government, as well as a carefully designed collaborative innovation trajectory.

4) **Integrated health care**: Organizing health care and social services for elderly people requires close cooperation between various health care, welfare, social and housing organizations.Moreover, providing tailor-made services in (health) care requires interdisciplinary collaboration in teams of professions.

Furthermore, many stakeholders are involved in these complex issues. Therefore, it may be difficult to deal with these challenges and a chaotic process may ensue, including unexpected and unwanted outcomes. Moreover, whilst handling these issues conflicts could occur, which cannot easily be resolved.

The complex nature of these challenges is caused by the lack of information or knowledge, or due to technological difficulties. In addition, the presence of various stakeholders, with diverging or even conflicting interests and perceptions, can result in a complex issue for which it may be difficult to find a solution. In addition, another feature of such issues is that they cut across the traditional jurisdiction of organizations. Moreover, it can be unclear which government layer (local, regional, national), is responsible for these issues. Furthermore, they can blur the traditional boundaries between public, private and societal domains.

Governments, businesses and civil society are often unable to tackle these issues themselves because they lack the resources or problem-solving capacities. In these cases, traditional methods of dealing with problems, policymaking and public service delivery, no longer suffice. Therefore, these complex challenges require a shift from a more traditional top-down way of problem-solving to a more horizontal collaborative way of governing and managing.

The statement “complex business problems are difficult to solve” is so apparent that it doesn’t require any justification.

Most complex business problems share the following characteristics:

- The **number of possible solutions** is so huge that it almost rules out a complete search for the best possible answer.
- The problem exists in **time-changing environment**. This means that decisions made in the past, found however optimal, might be actually far from optimality in present situations.
- The problem is **severelyconstrained**. The final solution for most of the problems have to satisfy various conditions imposed by organizational policies. Quite frequently finding even a single acceptable solution – the one satisfying all the constrains – is quite difficult.
- There exist **multiple objectives** of conflicting interest.

Let us discuss all the above primary characteristics in turn.

8.2 NUMBER OF POSSIBLE SOLUTIONS

An event manager is given an assignment of arranging a pizza party. He approached a Pizza commercial and asked for menu. Instead of presenting the menu to him, a pizza commercial asked him to choose from the following ingredients.

1. Choose 1 from 4 types of crust
2. Choose 1 from 4 sizes
3. Choose 1 from 5 types of cheese
4. Choose 1 from 4 types of sauce
5. Choose 1 from 3 levels of sauce
6. Choose up to 9 from available 9 types of non-veg content
7. Choose up to 15 from available 15 types of vegetarian content.

Pizza commercial claimed that anyone can combine the above ingredients to get more than 34 million different combinations. The event manager didn't believe it at all!

Let us help event manager to get the total number of different combinations to dispel his disbelief.

Crust can be chosen in 4 different ways, size can be chosen in 4 different ways, cheese in 5 different ways, types of sauce in 4 different ways, levels of sauce in 3 different ways.

Now, the types of non-veg content can be chosen in 2^9 ways and types of veg content in 2^{15} ways.

Thus, the total number of different combinations is

$$4 \times 4 \times 5 \times 4 \times 3 \times 29 \times 215 = 16{,}10{,}61{,}27{,}306$$

which is whoppingly larger than 34 million claimed by the modest people at pizza commercial.

As getting a menu printed for all these combinations is not feasible for any business, hence the Pizza commercial had only enlisted the ingredients in various categories and had given ample choices to the customer.

Let us consider another example, suppose a tourist has a list of cities that he plans to visit. He wants to go to each city exactly once. Naturally, he wants to tour all the cities in that order which requires travelling the shortest distance. Suppose that he is visiting four cities, labeled as A, B, C, D. He always starts and ends with city A, hishometown. Following are the distances between the cities: dist(A, B) =10, dist(A,C)=10, dist(A,D)=14, dist(B,C)=14, dist(B,D)=10, and dist(C,D)=10. In this case the distances are symmetric, i.e., dist(p, q)=dist(q, p)

Here are all the possible tours of the cities beginning from hometown:

1) A to B to C to D and back to A; total distance = 10 + 14 + 10 + 14 = 48; we can write this tour compactly as A – B – C – D – A
2) A to B to D to C and back to A; total distance = 10+10+10+10=40; we can write this tour compactly as A – B – D – C – A
3) A to C to B to D and back to A; total distance = 10 + 14 + 10 + 14 = 48; compactly written as A – C – B – D – A
4) A to C to D to B and back to A; total distance = 10+10+10+10=40; compactly written as A – C – D – B – A
5) A to D to B to C and back to A; total distance = 14 + 10 + 14 + 10 = 48; compactly written as A – D – B – C – A
6) A to D to C to B and back to A; total distance = 14 + 10 + 14 + 10 = 48; compactly written as A – D – C – B – A

Clearly some tours are shorter than the others. If we assume that these four cities are situated on the four vertices of a quadrilateral $\square$ABDC then shortest way to visit all the cities is to traverse along the perimeter of the quadrilateral and avoid the diagonals. By the by let us observe that total number of possible tours = 6=1 x 2 x 3 = 3! = (4-1)!, Incidentally 4 is the number or cities in our journey.

But not all instances of such tours are so straight-forward simple or so small in number.

For example, suppose we want to visit all 535 cities in Maharashtra state, it is not at all obvious how we can we go about finding the shortest possible tour of so many cities (which happens to be unbelievably large number, 535!), to top it all think if one plans to visit all 4000 cities in India, the number becomes incomprehensibly large not only for human mind but also is super challenge for super computers to compute.

Just to give you the feeling of awesomeness of these big numbers, consider 100! which is equal to

93326215443944152681699238856266700490715968264381621468592963895217599993229915608941463976156518286253697920827223758251185210916864000000000000000000000000= 9.332621544394415268169923885626670049071596826438162146859 2 × 10^{157}, comprising of 158 digits.

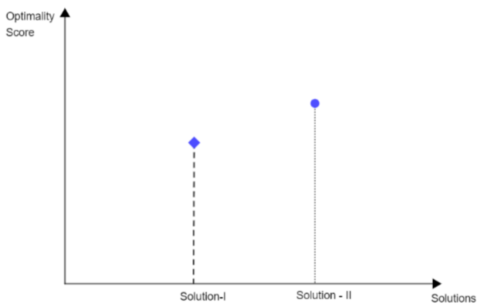
8.3 TIME-CHANGING ENVIRONMENT

Despite knowing very well that in the present complex times market-place is always in a flux, managerial personnel often take short term view to fix up the problems. But market-place is always dynamic accordingly these “static snapshots” of the problems and their quick fix solutions do not work optimally for any organization. Thus, real-world business problems being almost always set-in time-changing environment is pertinent to factor the time aspect also in working out possible solutions.

To illustrate this point, let us consider distribution of perishable commodities like fresh fruits and/or vegetables. Consider a firm, say in Nasik, producing a very large number of different types of fruits which it has planned to distribute all over country. Now, the very different time-zones, road conditions, vehicular density in different geographical locations are the contributing factors for the overall success in timely distribution of the commodities. Add to it the occasional labour strikes or delivery vehicles accidents, the problem becomes more compounded and its solution calls for addressing the time factor explicitly.

Supposing all considerations are thought over and a few possible solutions are proposed, for example, we have to decide to implement solution – I or solution – II.

Question arises as to which one of these solutions should be selected?



Apparently, it seems obvious that solution - II has a more optimality score, accordingly solution – II is better than solution – I. undoubtedly it is indeed true that solution – II is superior to solution – I but the reality may not be that straight-forward, as it appears.

It’s quite possible that solution – I has more stability, consistency, robustness whereas solution – II may be subjected to high sensitivity.

8.4 PROBLEM-SPECIFIC CONSTRAINTS

In context of Project Management, Work breakdown structure (WBS) is a method for getting a complex, multi-step project successfully implemented/executed. One of its aspect is to carry out various modules in time-bound manner. Clients often impose **‘Start No Earlier Than’** constraints due to their own schedules.

A schedule constraint is a condition placed on a project schedule that greatly influences the beginning and end date of a business project. These schedule constraints can take the form of fixed imposed dates for anactivity in WBS. Typically, a constraint date can be used to restrict the start of an activity to occur no earlier than a specified date, rather than allowing the dates to be determined by the planning process. For example, a publisher can't release a printed version of the book in the market until the printer can get its copies to the warehouse. Here is another example from construction busines. When a company lays the foundation of the building, it can't begin pouring concrete until the first day, a truck carrying cement arrives to deliver it.

To bring the clarity about this temporal complex, consider the tough task of preparing time-table of a school - an educational organization. First and foremost, we have to prepare a list of all the subjects along with their syllabi requirements (theory, practicals, activities, assignments, projects, etc.) Next, we need a database of all available teachers, together with their academic and other qualifications (professional, extra-curricular, skill-based, etc.). Immediately afterwards, we have to allocate the teaching workload to the respective teachers in the **best** possible way. Here by **best** we mean proper mapping of a teacher's qualification and the subject to be taught by him/her, in addition to, making optimal use of the teacher's other capabilities like leadership qualities, motivating ability, their inclination to social work, all of which contribute to not only the success of the school but also contribute to the nation building by producing well rounded personalities. We note however, that all the demands of allocating teachers as academic resources to various activities have to satisfy a few ***hard constraints***such as***:***

- A teacher cannot be physically present in two classrooms at one and same time.
- A teacher allotted specific subjects; activities must have requisite qualifications as per the laws of the educational bodies.
- This refers to 'Practicals' module of a given course. The number of students in a batch is subject to the capacity of Science Laboratory, number of lab assistants available therein.

Above illustrations are all about hard constraints, which a feasible solution (acceptable timetable) cannot at all violate. Besides these ***hard constraints***, there are many ***soft constraints*** as well. These are not very essential requirements but they do contribute to overall smooth functioning of the organization and thus, they are 'nice to have'.

- Providing first lecture of the day of any class to their respective class-teacher.
- Scheduling no more than three consecutive lectures on any given day for any teacher.
- Avoiding the outdoor sessions of physical education immediately before or after science practicals.

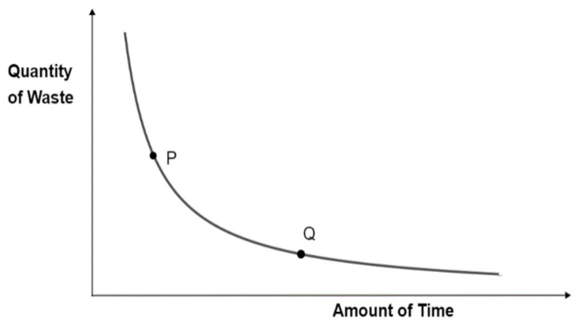
Considerations described above are more or less applicable to even transportation problems: constraints involving capacity limit (load carrying), delivery time windows, maximum driving time, deadline restrictions. A few of these constraints are clearly hard, as for example not loading chemicals with food items in the same compartment of the vehicle, whereas others are soft, personal preferences of drivers as regards to choose of vehicles available.

Irrespective of the problem under consideration whether school timetable or transportation of goods, it is essential to assign numeric weights to each soft constraint to emphasize their relative importance. In this way, while working out the possible solutions to a problem these weights can be used to compute final quality measure score for each possible solution.

8.5 MULTI-OBJECTIVE PROBLEMS

Seldom (quite infrequently) it happens that any actual problem from business world pertains to one objective. Generally, any business problem involves some or the other form of cost-benefit analysis which of course is of paramount importance in so far monetary aspects are concerned. However, in this compact term ‘cost-benefit’ analysis lie hidden various other issues as objectives which are important for optimizing production. These objectives often include, obviously, minimization of not only cost and production time but also minimization of material waste. That’s where the conflict factor creeps in. It is well-known observation that often the quantitative aspects of waste and time are inversely proportional. We note, in passing that these objectives might work against each other. That is, minimizing time of production often results in increase in material waste and vice-versa.

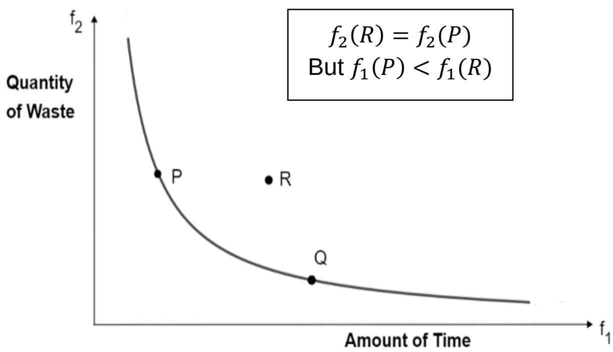
To illustrate the situation graphically, consider the bi-objective problem mentioned above. It is seen from the diagram the geometrical curve represents approximate relationship between the objectives of minimizing quality of waste and minimizing amount of time.



Now, we consider the solution points P and Q. To decide as to which one is better, is not that easy. Apparently, solution P is quite faster (i.e., fulfills the objective of minimizing time), however, the quantity of material waste is quite higher and conversely. In general, when we deal with multi-objective problems, its quite possible to get a solution which is best from point of view of first objective but not from viewpoint of other objectives, and there could be altogether a different solution which may be best with respective to second objective but not the first. Thus, we are posed with the challenge of selecting the optimum solution of multi-objective problems by linking quality of solution with various (probably conflicting) parameters. Thus, our question, as to which of the solutions P or Q be selected still remains unanswered, until one arrives at common consensus on issues of time and waste.

Concept of **multi-objective optimization problem(MOP)** is also applicable to real world-version of travelling salesman problem as well. In addition to minimizing the distance of total journey, it is also desirable to go on fulfilling the requirement of “timely” deliveries on the way. Apart from these objectives, one has to consider the overall travel time among the delivery trucks, a few of those vehicles may carry preprocessed goods which are to arrive at specific plant for further processing, after which some other vehicles from thereat have to deliver it to “packaging department” for ready-to-sell goods.Finally, these packaged goods get disbursed by distribution network agency.

However, there may arise possibilities of many ***candidate solutions*** to MOP under consideration, in which case, it is quite convenient to categorize these candidate solutions into ***dominated*** and ***non-dominated*** solutions.



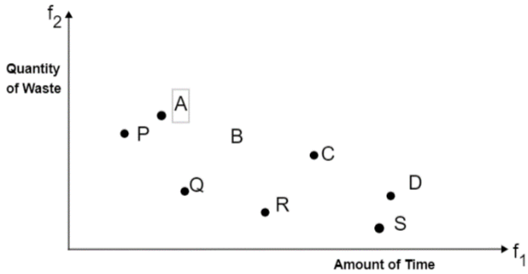
In what follows, f_r denotes the objective function with rank r, among n such objective functions $f_1, f_2, ... f_n$ for a given MOP.

A solution $x = (x_1, x_2, ..., x_n)$to an MOP is said to dominate a solution $y = (y_1, y_2, ..., y_n)$ if and only if $f_i(x) \leq f_i(y)$for $i = 1, 2, ..., m,$ and there exists at least one $j(1 \leq j \leq n)$ such that $f_i(x) < f_i(y)$

With reference to above diagram, solution at point P dominates that at R, alternatively we say solution R is dominated since solution at P is no doubt as good as solution at R on the quantity of waste factor and better on the time factor.

Clearly, a candidate solution which is not dominated by any other feasible solution is called a ***non-dominated***.

With reference to the adjacent diagrammatic illustration, solutions at point A, B, C, D are dominated:



1) As discussed earlier, solution P dominates solution A, since P scores better on all objectives, i.e., solution A is dominated.
2) Solution Q dominates solution B, as Q is better on all objectives.
3) Solutions C and D are dominated by solution R.

Thus, we conclude, solutions at points P, Q, R S are non-dominated since there do not exists any other solution, considering all objectives, which are as good as or better than any of these.

Concept of dominance is there to differentiate the dominated and non-dominated solution. Of course, it is only non-dominated solution which matter to us. Almost each and every such solution (obtained by solving MOP concerned with any system) could be of some interest to us in one way or the other but often, we can only implement one solution. This happens because the non-dominated solution and value in MOP are usually achieved when one objective function cannot increase without reducing the other objective function.

Finally, in order to decide the most practical solution, among the various non-dominated solutions, very often either human-expertise is called for to select economic trade-off or some higher-level knowledge needs to be applied.

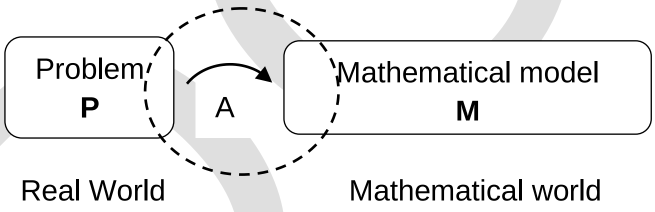
As an illustration, we may note down the comparative importance of each objective function either by assigning numeric weights or by allotting proper ranking to all objectives and thereafter choosing the solutions in accordance with this ranking. Another approach is to select the most important objective function and formulate the remaining ones into constraints that need to be satisfied. Ultimately, the chosen approach depends upon MOP under consideration.

8.6 MODELING THE PROBLEM

Broadly speaking, any problem-solving activity is performed in two stages (1) developing a model of the problem, and (2) generating feasible solution using that model.

A mathematical model, put succinctly, is a mathematical relation that describes some real-life situations.

The aim of mathematical modelling is to get some useful information about a real-world problem by converting it into a mathematical problem.



However, this way of solving real-world problems demands due attention to details, in the sense that here we are obtaining solution to the developed *model*of the problem. Depending on the accuracy with which the model had been worked out, solution may turn out to be meaningful or vague or meaningless.

The process of constructing mathematical models, is called mathematical modeling. The various stages involved in mathematical modeling are

1) **Formulation:** Having picked a real-world problem, it is expressed in equivalent mathematical form.
2) **Solution:**The formulated problem is then solved using appropriate mathematical tools and techniques.
3) **Interpretation:** The solution, thus obtained is interpreted in-terms of real-world problem.
4) **Validation:**validation refers tothe extent to which the solution is applicable in context of real-world situation.

One may recall, as to how these aspects of mathematical modeling, are carried out, has been discussed while solving linear programming problems.

When we are solving a real-life problem, formulation can require a lot of time.

1) **Formulation** involves the following three steps:

(i) **Stating the problem:** Often the real-world problem is stated ambiguously. To overcome this, we may need to redefine it many times.

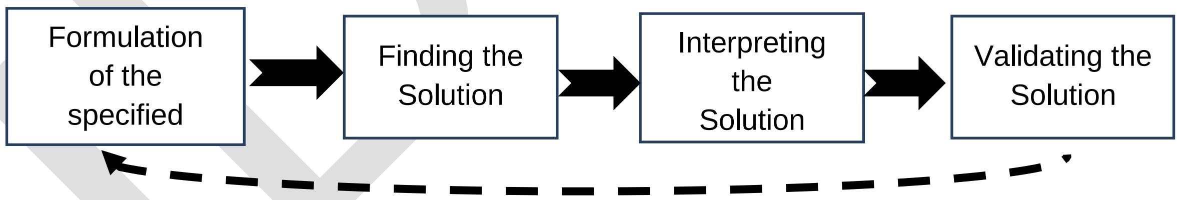
(ii) **Identifying the relevant factors:** Deicide which quantities and relationships are important for our problem and which are unimportant and can be discarded.

(iii) **Mathematical Description:** Now, suppose we are clear about what the problem is and what aspects of it are more relevant than the others. Then we have to find a relationship between the aspects involved in the form of an equation, a graph or any other suitable mathematical description. If it is an equation, then every important aspect should be represented by a variable in our mathematical equation.

2) **Finding the solution**: Just formulating the problem mathematically does not yield the solution. The mathematical equivalent of the problem is to be solved using various techniques from different branches of mathematics.

3) **Interpreting the solution**: The solution is value of some mathematical expression or values of the decision variables in the model. We have to interpret these values in context to real-world problem and understand the significance of these values.

4) **Validating the solution**:After getting the solution and interpreting it, weneed to check whether the solution as it is, can be implemented practically. If so, then the mathematical model is acceptable. If the solution is not feasible, we go back and reformulatethe model.



Once a real-life problem is solved thus, then scaled-up models can be constructed. These scaled-up models may provide either exact solutions or approximate.

Limitations of Modeling:

Sometimes such mathematical modeling turns out to be of very limited usefulness. Many a times we have to be satisfied even with approximate solutions. For example, while modelling the weather phenomena, the differential equations involved are so complex to solve that we may use numerical methods to find an approximate solution and accept it.

We may ask to what extent can the model be improved. Often, to improve it, we may have to consider more factors which may not be apparent initially. While doing this, we introduce more variables in mathematical formulation. All this may result in a very complicated model which may be quite difficult to use and solve.

Theoretically as well as practically, it is desirable that a model of the real-world problem must be simple to use and solve. A good model must have following two characteristics: (1) Ease of use (2) Accuracy, how closely it matches with real problem.

So is mathematical modeling a universal solution to all our problems? The answer is emphatic NO. Let's see the reason for it. Often, the objective function (profit function, cost function, production function etc.) in many models turn out to be discontinuous and the discontinuities pose severe difficulties for usual techniques to find optimal solutions. Thus, the solutions we often get by applying traditional techniques (based on concept of gradient) on these functions turn out to be of poor quality. Even though in some cases, we may get exact solution which is practically very difficult to implement. Thus, even though our model might be perfect, it becomes useless to take decisions. In such cases, the further course of action taken is to try to simplify the model in order that traditional optimization methods can be used to produce superior solutions or leaving the model unchanged and applying non-traditional approach to obtain near optimum solution. Simply put, the first course of action essentially used approximate model of a real-world problem and thereafter find the precise solution, whereas second course of action carries out with precise model of real-world problem and obtains approximate solution.

Even though first course of action appears to be best, but it is not so! Essentially this precise solution is in context of simplified model and not of the actual real-world problem! Second course of action is superior. To understand why it is so, we need to look at it from this point of view. The approximate model of first approach ends up hiding the inherent irregularities of a real-world problem. Thereby making provision for traditional methods to give us precise solutions, which is not at all optimal as much information is lost by hiding the said irregularities. However, the second approach of developing precise model subjected to non-traditional methods for solving the problem is quite faithful to reality even though it yields an approximate solution which can very well serve the purpose of optimal solutions from all practical considerations.

8.7 REAL – WORLDEXAMPLE

Outline and Methodology to approach and resolve the problem.

Due to industrial revolution that began in 18th century, the ubiquitous problem faced by every country is that of pollution due to chemical emissions. To overcome or reduce the pollution level without compromising on the economic development and infrastructure demands of country/ state/ city is very daunting task due to tremendous complexity of huge number of dynamically changing variables related to large number of issues involved. The issues concern not only with various departments (state pollution control board, environmental ministry, industrial development governing bodies, economic advisories council, local governing bodies, etc.) but also collecting tremendous amount of data sets in varying geographical locations at different times during the period of survey to compile the relevant data in order to understand the situation and problem better. This itself is indication regarding unbelievably large number of variables to be considered, restrictions imposed resulting in huge number of constraints. The brief discussion so far clearly conveys that we are dealing with MOP, in which apparently a few objective functions are clearly of conflicting interest and we have to come out with the optimal solution – satisfying as many constraints as possible without ignoring the larger issues of human health and economic development of the land.

Before we get under the skin of the problem, let us consider typical case of 'Air Pollution' in any state: Air pollution is caused mainly by transportation, fuel combustion in stationary sources, burning of fossil fuels like coal, wood, dry grass, and construction activities. Motor vehicles produce high levels of carbon monoxide (CO), hydrocarbons (HC) and nitrogen oxides (NO). Construction activities, bad roads and burning of fossil fuels are responsible for dust (particulate matter) pollution. Residential and commercial activities also contribute to air pollution. Human health is affected due to poor air quality. Principally, air pollution affects the body's respiratory system and the cardiovascular system. Though the individual reactions to air pollutants depend on the type of pollutant a person is exposed to and the degree of exposure, air pollution may cause long term health problems. The health effects caused by air pollutants may range from biochemical and physiological changes like difficulty in breathing, wheezing, coughing and aggravation of existing respiratory and cardiac conditions.

The above-mentioned details clearly bring out the complexity of MOP under consideration and that too just from one aspect of pollution!As such there is need for carefully planning outthe methodology to understand and to workout the best solution of the MOP and chalk out further strategies.

The first step in such studies is to visualize the geometry of the land using surprisingly traditional methods of dividing the entire region into square grids. The number and type of chemical factories producing air pollutants, pollution density, vehicular density, wind conditions, etc., in the partitioned grids are taken note of and then assigned numeric weights after consulting the experts in these matters. The medical records of people suffering from respiratory diseases caused basically due to air pollution are then compiled zone-by-zone. Even the data regarding green environment (number of trees in and around residential areas, gardens, number of trees on highways, etc.) is then compiled. Keeping the guidelines and objectives of various institutes, departments, bodies involved, multiple objectives are figured out and constraints are put forthwith. This is how an initial model is formulated/designed. To find the feasible solutions, as discussed earlier, traditional or non-traditional mathematical techniques are considered. Applying the chosen approach, depending upon the relative importance of multiple objectives, feasible solutions are worked out, interpreted, and after validating thus found solutions, they are then implemented in the best possible manner. Even these initial solutions may require some tweaking and are also subjected to periodic checks because the entire MOP system is dynamic (spatially as well as temporally). Having put the system in place, we cannot afford to be complacent due to the constantly changing parameters affecting the data variables and changing priorities which affect the relative importance of multiple objectives. This calls for constant supervision, observations of type 'before' and 'after', to bring about modifications and hence the need for 'evolutionary algorithm automated systems', which work on the philosophy of modern times: "Today's optimal solution may become obsolete tomorrow."

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UNIT 7: Chapter 9

MULTIPLE ATTRIBUTE DECISION MAKING AND MULTIPLE CRITERIA DECISION MAKING

Unit Structure

9.0 Objective

9.1 Introduction

9.2 Difference between MADM and MCDM

9.3 Multiple Attribute Decision Making(MADM)

- 9.3.1 Analytic Hierarchy Process (AHP)
- 9.3.2Weighted Sum Model (WSM)
- 9.3.3Simple Additive Weighing(SAW)

9.4 Multiple Criteria Decision Making(MADM)

9.4.1 Weighted Product model(WPM)

- 9.4.2 Entropy: Shanon Theory
- 9.4.3 VIKOR Method

9.0 OBJECTIVES

After going through this unit, you will be able to:

- formulate the decision making problem
- solve the decision making problem using the appropriate method.

9.1 INTRODUCTION

In the decision making problem there are attribute and criteria. The decision is to be taken when there are many attribute or many criteria. There are quantitative and qualitative problems. Considering all these factors the decision is to be taken.

There are two types of decision making problems viz Multiple Attribute Decision Making and Multiple Criteria Decision Making. Some methods to solve these these problems are studied in this unit.

9.2DIFFERENCE BETWEEN MADMAND MCDM

The differences are given below.

Multiple Attribute Decision Making MADM		Multiple Criteria Decision Making MCDM	
1.	Refers to Multi Attribute Decision Making.	1.	Refers to Multi Criteria Decision Making.
2.	Makes decision based on limited number of pre-determined alternatives.	2.	Makes decision based on multiple usually conflicting criteria.
3.	MADM concerns objects that can be described through several attributes.	3.	Uses several criteria to rank objects.
4.	MADM specifies how attribute information is to be processed in order to arrive at a choice.	4.	MCDM method specifies how criteria could be ranked in order to arrive at a choice.
5.	Used for making decision in qualitative problem.	5.	Used for making decision in quantitative as well as qualitative problem.
6.	MADM involves SAW, WPM, AHP.	6.	MCDM involves MODM & MADM based on the problem whether it is selection problem or design problem.
7.	Eg.: When buying a car, we could take criteria of acceleration with attributes like car weight, horsepower, shape of car.	7.	Eg.: When buying a car, we could take criteria such as price, horsepower.

9.3 MULTIPLE ATTRIBUTE DECISION MAKING (MADM)

In this type of problems, there are multipleattributes. We want to take decision. There are many methods of decision making in this case. These methods are discussed below.

9.3.1ANALYTIC HIERARCHY PROCESS(AHP)

- Developed by "Tom Saaty".
- Decision making method for prioritizing alternatives when multi-criteria must be considered.
- Structuring a problem as a hierarchy or set of integrated levels.
- Structured in 3 levels:
    - Goal : Best car to purchase
    - Criteria : Cost, safety and appearance
    - Alternatives : Car themselves
- AHP never requires absolute judgment /assessment.
- It requires you to make a relative assessment between two items at a time.
- AHP judgments are called pair-wise comparisons.
- Uses a weighted average approach idea, but it uses a method for assigning rating and weights that are considered more reliable and consistent.

Phases in AHP:

1. Decompose a problem into hierarchy.
    - Find criteria
2. Collect I/p data by pair-wise comparisons of criteria at each level of the hierarchy/alternatives.
3. Estimate relative importance (weights) of criteria & alternatives & check consistency in pair-wise comparisons.
4. Aggregate the relative weights of criteria & alternatives to obtain a global ranking of each alternative with regards to the goal.

Example 7.1: The board of directors have to choose a leader for a company whose founder is about to retire. There are three competing candidates Mr. Adwik, Mr. Vedant, Miss.Smahi and four competing criteria, Experience, Education, Charisma, and Age. Use AHP to choose the most suitable candidate.

The comparison matrix forpair wise criteria is given below.

Criteria	Experience	Education	Charisma	Age
Experience	1	4	3	7
Education	1/4	1	1/3	3
Charisma	1/3	3	1	5
Age	1/7	1/3	1/5	1

Also the relative criteria for alternatives are:

Experience	Mr. Adwik	Mr. Vedant	Miss. Smahi
Mr. Adwik	1	1/4	4

Mr. Vedant	4	1	9
Miss. Smahi	1/4	1/9	1

Education	Mr. Adwik	Mr. Vedant	Miss. Smahi
Mr. Adwik	1	3	1/5
Mr. Vedant	1/3	1	1/7
Miss. Smahi	5	7	1

Charisma	Mr. Adwik	Mr. Vedant	Miss. Smahi
Mr. Adwik	1	5	9
Mr. Vedant	1/5	1	4
Miss. Smahi	1/9	1/4	1

Age	Mr. Adwik	Mr. Vedant	Miss. Smahi
Mr. Adwik	1	1/3	5
Mr. Vedant	3	1	9
Miss. Smahi	1/5	1/9	1

<u>Solution</u>: Step 1) Find weights for Relative (Criteria Vs Criteria)

Criteria	**Experience**	**Education**	**Charisma**	**Age**
Experience	1	4	3	7
Education	1/4	1	1/3	3
Charisma	1/3	3	1	5
Age	1/7	1/3	1/5	1
Sum	**1.726**	**8.333**	**4.533**	**16**

Divide every element by column sum and then take row average.

Criteria	**Experience**	**Education**	**Charisma**	**Age**
Experience	0.579	0.48	0.662	0.54
Education	0.145	0.12	0.074	0.132
Charisma	0.193	0.36	0.221	0.272
Age	0.083	0.04	0.044	0.058

Step 2) Find weights of each of the criteria (Alternative Vs Alternative)

Experience	Mr. Adwik	Mr. Vedant	Miss. Smahi

Mr. Adwik	1	1/4	4
Mr. Vedant	4	1	9
Miss. Smahi	1/4	1/9	1
Sum	5.25	1.361	14

Divide every element by column sum and then take row average.

Experience	Mr. Adwik	Mr. Vedant	Miss. Smahi	Row average weight or eigen value
Mr. Adwik	0.19	0.184	0.286	**0.22**
Mr. Vedant	0.762	0.735	0.643	**0.713**
Miss. Smahi	0.048	0.082	0.071	**0.067**

Similarly the row average weights for education, cherishma and age are

Education	Mr. Adwik	Mr. Vedant	Miss. Smahi	Row average weight or eigen value
Mr. Adwik	1.158	0.273	0.149	**0.193**
Mr. Vedant	0.053	0.091	0.106	**0.083**
Miss. Smahi	0.79	0.636	0.745	**0.724**

Cherisma	Mr. Adwik	Mr. Vedant	Miss. Smahi	Row average weight or eigen value
Mr. Adwik	0.763	0.8	0.643	**0.735**
Mr. Vedant	0.153	0.16	0.286	**0.2**
Miss. Smahi	0.085	0.04	0.071	**0.065**

Age	Mr. Adwik	Mr. Vedant	Miss. Smahi	Row average weight or eigen value
Mr. Adwik	0.238	0.231	0.333	**0.267**
Mr. Vedant	0.714	0.693	0.6	**0.669**
Miss. Smahi	0.048	0.077	0.067	**0.064**

Step 3) The composite impact table

Weights	0.54	0.132	0.272	0.058

Criteria→	Experience	Education	Charisma	Age
Mr. Adwik	**0.22**	**0.193**	**0.735**	**0.267**
Mr. Vedant	**0.713**	**0.083**	**0.2**	**0.669**
Miss. Smahi	**0.067**	**0.724**	**0.065**	**0.064**

Composite impact of Mr. Adwik = 0.54×0.22+0.132×0.193+0.272×0.735+0.058×0.267
= 0.359

Composite impact of Mr. Vedant = 0.54×0.713+0.132×0.083+0.272×0.2+0.058×0.669
= **0.489**

Composite impact of Miss. Smahi= 0.54×0.067+0.132×0.724+0.272×0.065+0.058×0.064
= 0.153

Best composite score is 0.489

Hence best alternative is choose Mr. Vedant

9.3.2 WEIGHTED SUM MODEL (WSM)

- Weighted Sum Model - Simple MCDM method.
- It is evaluating a number of alternatives in term of number of decision criteria.
- It is applicable only when all the data are expressed in **exactly same unit**.
- Total importance of alternative Ai , denoted as ,A;

$$\mathrm{Ai}(\mathrm{wsm\text{-}score}) = \sum W_j a_{ij} \quad \text{for } i = 1,2,3,\cdots,m$$

Example 7.2:Suppose that an MCDM problem involves four criteria, which are expressed in the same unit, and three alternatives. The relative weights of the four criteria were determined to be: w_1=0.20, w_2=0.15, w_3=0.40 and w_4=0.25. The corresponding a_{ij} values are assumed to be as follows:

A	25	20	15	30
	10	30	20	30
	30	10	25	10

Find the best alternative using the weighted sum method (WSM).

Solution: Let the alternatives be: A_1, A_2 and A_3
and the criteria be: C_1, C_2, C_3 and C_4

Given

Weights	0.20	0.15	0.40	0.25
	C_1	C_2	C_3	C_4
A_1	25	20	15	30

A_2	10	30	20	30
A_3	30	10	25	10

According to WSM the best alternative (in maximization case) is indicated by the following relationship:

$$A^{*}_{WSM} = \max_{i} \sum_{j=1}^{N} q_{ij} w_j \quad \text{for } j=1,2,3, ---, N$$

A_1(WSM score) = 25×0.20+20×0.15+15×0.40+30×0.25 = 21.50

A_2(WSM score) = 10×0.20+30×0.15+20×0.40+30×0.25 = 22.00

A_3(WSM score) = 30×0.20+10×0.15+25×0.40+10×0.25 = 20.00

Therefore the best alternative is A2.

9.3.3 SIMPLE ADDITIVE WEIGHTING (SAW)

- Simplest and widest used method.
- Each attribute is given a weight and sum of all weight must be 1.
- Each alternative is assessed with respect to every attribute. It is used only when decision attribute can be expressed in identical units of measure.
- If all the attribute of decision table are normalize then SAW can be used for any type and any number of attribute.
- Pi $= \sum_{j=1}^{m} wj * (Mij)$normal
- Alternative with high value of Pi is the best alternative.
- It is a proportional linear transformation of raw data which means relative order of magnitude of standard score remains equal.

<u>Example 7.3</u>: Use SAW method to suggest the best alternatives? Where C_1, C_2, C_3, C_5, $C_6$are beneficiary criteria and C_4 isnon beneficiary criteria.

Weight	0.2	0.1	0.1	0.1	0.2	0.3
	C_1	C_2	C_3	C_4	C_5	C_6
A_1	2.0	1500	20000	5.5	5	9
A_2	2.5	2700	18000	6.5	3	5
A_3	1.8	2000	21000	4.5	7	7

A_4	2.2	1800	20000	5.0	5	5

Solution:

For max Benefit column=current value/largest value

For min Benefit column=min value/current value

For given attributes

C_1,C_2,C_3,C_5,C_6 are max beneficial column and C_4 is non-beneficial or negative column.

For C_1 max=2.5, C_2 max=2700, C_3 max=21000, C_5 max=7, C_6 max=9

For C_4 min=4.5

Weight	0.2	0.1	0.1	0.1	0.2	0.3
	C_1	C_2	C_3	C_4	C_5	C_6
A_1	2.0/2.5	1500/2700	20000/21000	4.5/5.5	5/7	9/9
A_2	2.5/2.5	2700/2700	18000/21000	4.5/6.5	3/7	5/9
A_3	1.8/2.5	2000/2700	21000/21000	4.5/4.5	7/7	7/9
A_4	2.2/2.5	1800/2700	20000/21000	4.5/5.0	5/7	5/9

Normalized Matrix is given below:

Weight	0.2	0.1	0.1	0.1	0.2	0.3
	C_1	C_2	C_3	C_4	C_5	C_6
A_1	0.8	0.56	0.95	0.82	0.71	1.0
A_2	1.0	1.0	0.86	0.69	0.43	0.56
A_3	0.72	0.74	1.0	1.0	1.0	0.78
A_4	0.88	0.67	0.95	0.9	0.71	0.56

The performance score for each alternative is calculated as follows:
A_1=(0.8)(0.2)+(0.56)(0.1)+(0.95)(0.1)+(0.82)(0.1)+(0.71)(0.2)+(1.0)(0.3) = 0.835

Similarly,

A_2=0.709, A_3=0.852, A_4=0.738,

Hence A_3>A_1>A_4>A_2

Thus highest performance score A_3 is selected.

9.4 MULTIPLE CRITERIA DECISION MAKING (MCDM)

In this type of problems, there are multiple criteria. We want to take decision. There are many methods of decision making in this case. These methods are discussed below.

9.4.1WEIGHTED PRODUCT MODEL (WPM)

- Weighted product model(WPM) is a popular multi criteria decision analysis/making model. It is similar to weighted sum model.
- Difference is instead of addition there is multiplication in WPM.
- As with all MCDA/MCDM methods given is a finite set of decision alternatives described in terms of decision criteria.
- Each decision alternatives is compared with others by multiplying a number of ratios one for each decision criteria.
- Each ratio is raised to the power equivalent to the relative weight of the corresponding criteria.
- Suppose MCDM problem has ‘m’ alternatives and ‘n’ decision criteria.
- **Assuming that all criteria are benefit criteria i.e higher values are better.**

W_j→ relative weight of criteria j
A_{ij}→performance value of A_i when it is evaluated in terms of C_j

If one wishes to compare the two alternatives
A_k& A_L (where m>=k & L>=1)
Then following product is calculated.

$$P\left(\frac{A_k}{A_L}\right) = \Pi_{j=1}^{m} \left(\frac{a_{kj}}{ai_j}\right)^{wj} \text{,for k,L=1,2,3,….m}$$

- If the ratio $P(A_k/A_L)$ is greater than or equal to value 1, then it indicates alternative A_k is more desirable than alternative A_L .
- WPM is often times called “dimensionless analysis” because its mathematical structure eliminates any units of measure.

<u>Example 7.4</u>:Suppose that an MCDM problem involves four criteria, which are expressed in the same unit, and three alternatives. The relative weights of the four criteria were determined to be: w_1=0.20, w_2=0.15, w_3=0.40 and w_4=0.25. The corresponding a_{ij} values are assumed to be as follows:

	25	20	15	30
A	10	30	20	30
	30	10	30	10

Find the best alternative using the weighted sum method (WPM).

<u>Solution</u>: Let the alternatives be: A_1, A_2 and A_3
and the criteria be: C_1, C_2, C_3 and C_4
Given

Weights	0.20	0.15	0.40	0.25
	C_1	C_2	C_3	C_4
A_1	25	20	15	30
A_2	10	30	20	30
A_3	30	10	30	10

Calculate the following products.

$$P\left(\frac{A_k}{A_L}\right) = \Pi_{j=1}^{m}\left(\frac{a_{kj}}{ai_j}\right)^{wj} \text{,for k,L=1,2,3,….m}$$

$$P\left(\frac{A_1}{A_2}\right) = \left(\frac{25}{10}\right)^{0.20} \times \left(\frac{20}{30}\right)^{0.15} \times \left(\frac{15}{20}\right)^{0.40} \times \left(\frac{30}{30}\right)^{0.25} = 1.007 > 1$$

Similarly;

$$P\left(\frac{A_1}{A_3}\right) = \left(\frac{25}{30}\right)^{0.20} \times \left(\frac{20}{10}\right)^{0.15} \times \left(\frac{15}{30}\right)^{0.40} \times \left(\frac{30}{10}\right)^{0.25} = 1.067 > 1$$

$$P\left(\frac{A_2}{A_3}\right) = \left(\frac{10}{30}\right)^{0.20} \times \left(\frac{30}{10}\right)^{0.15} \times \left(\frac{20}{30}\right)^{0.40} \times \left(\frac{30}{10}\right)^{0.25} = 1.059 > 1$$

The ranking of these alternatives follow: $A_1 > A_2 > A_3$.
Therefore, A_1 is best alternative.

<u>OR</u>

P(A1)=20.63, P(A2)=20.48, P(A3)=19.33.

Therefore A1 is the best alternative as value is more than any other option.

9.4.2 ENTROPY: SHANON THEORY

MADM models are selector models and used for evaluating, ranking and selecting the most appropriate alternative among attributes.

Entropy Method is used for calculating weights of the given matrix of alternative vs criteria.

Shannon's entropy is a well-known method in **obtaining the weights** for an MADM problem especially when obtaining a suitable weight based on the preferences and DM experiments are not possible.

<u>Example 7.5</u>: Calculate normalized decision matrix from the following.

	C_1	C_2	C_3	C_4
A_1	25	20	15	30
A_2	10	30	20	30
A_3	30	10	30	10

Also find the normalized weights of the attributes.

<u>Solution:</u>

STEP 1: Normalize the Weights

$$R_{ij} = \frac{x_{ij}}{\sum_{i=1}^{m} x_{ij}}$$

$$R_{11} = \frac{x_{11}}{\sum_{i=1}^{3} x_{1j}} = \frac{x_{11}}{x_{11}+x_{12}+x_{13}+x_{14}} = \frac{25}{25+20+15+30} = 0.277$$

$$R_{12} = \frac{x_{12}}{\sum_{i=1}^{3} x_{1j}} = \frac{x_{12}}{x_{11}+x_{12}+x_{13}+x_{14}} = \frac{20}{25+20+15+30} = 0.222$$

$$R_{13} = \frac{x_{13}}{\sum_{i=1}^{3} x_{1j}} = \frac{x_{13}}{x_{11}+x_{12}+x_{13}+x_{14}} = \frac{15}{25+20+15+30} = 0.167$$

$$R_{14} = \frac{x_{14}}{\sum_{i=1}^{3} x_{1j}} = \frac{x_{14}}{x_{11}+x_{12}+x_{13}+x_{14}} = \frac{30}{25+20+15+30} = 0.333$$

$$R_{21} = \frac{x_{21}}{\sum_{i=1}^{3} x_{2j}} = \frac{x_{21}}{x_{21}+x_{22}+x_{23}+x_{24}} = \frac{10}{10+30+20+30} = 0.111$$

$$R_{22}=\frac{x_{22}}{\sum_{i=1}^{3}x_{2j}}=\frac{x_{22}}{x_{21}+x_{22}+x_{23}+x_{24}}=\frac{30}{10+30+20+30}=0.333$$

$$R_{23}=\frac{x_{23}}{\sum_{i=1}^{3}x_{2j}}=\frac{x_{23}}{x_{21}+x_{22}+x_{23}+x_{24}}=\frac{20}{10+30+20+30}=0.222$$

$$R_{24}=\frac{x_{24}}{\sum_{i=1}^{3}x_{2j}}=\frac{x_{24}}{x_{21}+x_{22}+x_{23}+x_{24}}=\frac{30}{10+30+20+30}=0.333$$

$$R_{31}=\frac{x_{31}}{\sum_{i=1}^{3}x_{3j}}=\frac{x_{31}}{x_{31}+x_{32}+x_{33}+x_{34}}=\frac{30}{30+10+30+10}=0.375$$

$$R_{32}=\frac{x_{32}}{\sum_{i=1}^{3}x_{3j}}=\frac{x_{32}}{x_{31}+x_{32}+x_{33}+x_{34}}=\frac{10}{30+10+30+10}=0.125$$

$$R_{33}=\frac{x_{33}}{\sum_{i=1}^{3}x_{3j}}=\frac{x_{33}}{x_{31}+x_{32}+x_{33}+x_{34}}=\frac{30}{30+10+30+10}=0.375$$

$$R_{34}=\frac{x_{34}}{\sum_{i=1}^{3}x_{3j}}=\frac{x_{34}}{x_{31}+x_{32}+x_{33}+x_{34}}=\frac{10}{30+10+30+10}=0.125$$

The normalized matrix is as follows:

	C_1	C_2	C_3	C_4
A_1	0.277	0.222	0.167	0.333
A_2	0.111	0.333	0.222	0.333
A_2	0.375	0.125	0.375	0.125

STEP 2: The amount of decision information associated with each attribute can be measured by **entropy** value e_j isas follows:-

$$e_j=-k\sum_{i=1}^{N}R_{ij}\times\ln\left(R_{ij}\right)$$

Where $k = \frac{1}{\ln(N)}$ is a constant that guarantees $0 \le e_j \le 1$

Here $k = \frac{1}{\ln(3)} = 0.91$

$$e_1 = -0.91 \sum_{i=1}^{3} R_{i1} \times \ln(R_{i1})$$

$$e_1 = -0.91[R_{11} \times \ln(R_{11}) + R_{21} \times \ln(R_{21}) + R_{31} \times \ln(R_{31})]$$

$$e_1 = -0.91[0.277 \times \ln(0.277) + 0.111 \times \ln(0.111) + 0.375 \times \ln(0.375)] = 0.87$$

Similarly,

$$e_2 = -0.91[R_{12} \times \ln(R_{12}) + R_{22} \times \ln(R_{22}) + R_{32} \times \ln(R_{32})]$$

$$e_2 = -0.91[0.222 \times \ln(0.222) + 0.333 \times \ln(0.333) + 0.125 \times \ln(0.125)] = 0.87$$

$$e_3 = -0.91[R_{13} \times \ln(R_{13}) + R_{23} \times \ln(R_{23}) + R_{33} \times \ln(R_{33})]$$

$$e_3 = -0.91[0.167 \times \ln(0.167) + 0.222 \times \ln(0.222) + 0.375 \times \ln(0.375)] = 0.90$$

$$e_4 = -0.91[R_{14} \times \ln(R_{14}) + R_{24} \times \ln(R_{24}) + R_{34} \times \ln(R_{34})]$$

$$e_4 = -0.91[0.333 \times \ln(0.333) + 0.333 \times \ln(0.333) + 0.125 \times \ln(0.125)] = 0.90$$

STEP 3: To Calculate Degree of Diversification

$$d_j = 1 - e_j$$

$d_1 = 0.13$, $d_2 = 0.13$, $d_3 = 0.1$, $d_4 = 0.1$

STEP 4: To Calculate Weight (0.46)

$$w_j = \frac{d_j}{\sum_{k=1}^{m} d_k}$$

$w_1 = \frac{0.13}{0.46} = 0.28$, $w_2 = \frac{0.13}{0.46} = 0.28$, $w_3 = \frac{0.1}{0.46} = 0.22$, $w_4 = \frac{0.1}{0.46} = 0.22$

Note: Sum of weight should be equal to 1.

9.4.3 VIKOR METHOD

- The compromise ranking method, also known as VIKOR (VIšekriterijumskoKOmpromisnoRangiranje), was introduced as an applicable technique to implement within MADM.
- It is helpful when decision maker is not available.
- Provides max. group utility of majority and min. of individual regret of opponent.
- The compromise solution could be basis for negotiation involving the decision makers preference by attribute weights.

STEP 1: Identify Best and Worst value from the above constraints and determine objective and identify evaluation attributes.

STEP 2: Identify beneficial & non-beneficial attributes
Calculate values for E_i& F_i

For Beneficial attributes: $\frac{[(Mij)_{\max} - Mij]}{[(Mij)_{\max} - (Mij)_{\min}]}$

For Non-beneficial attributes: $\frac{[Mij - (Mij)_{\min}]}{[(Mij)_{\max} - (Mij)_{\min}]}$

Write the Decision Matrix (m_{ij}).

Weights $\{w_j\}$are known.

Find $E_i = \sum_j w_j \times m_{ij}$

$R_1 = \max(w_1 \times d_1, w_2 \times d_2, \ldots..)$

Calculate E_{max}, E_{min}, R_{max}, R_{min}

STEP 3: Find performance of each alternative (Pi)

$$P_i = v\left[\frac{Ei - E\min}{E\max - E\min}\right] + (1 - v)\left[\frac{Ri - R\min}{R\max - R\min}\right]$$

v = vikor strategy value (should be between 0 -1)
if value is not given assume 0.5

STEP 4: Arrange performance values in ascending order. Best alternatives ranked by Pi is the one with **minimum** value of Pi

Exercise 7:

1) Find the best alternative using Simple Additive Weighing (SAW) method.

Alternatives	Attributes					
	C_1	C_2	C_3	C_4	C_5	C_6
A_1	2.0	1500	20000	5.5	avg	very high
A_2	2.5	2700	18000	6.5	low	avg
A_3	1.8	2000	21000	4.5	high	high
A_4	2.2	1800	20000	5.0	avg	avg

Given attribute C_4 is non beneficiary and all other attributes are beneficiary.

2) Use SAW method to suggest the best alternatives? Where C_1 and C_3 are beneficiary criteria and C_2 and C_4 are non beneficiary criteria.

Weights	0.2	0.3	0.4	0.1
	C_1	C_2	C_3	C_4
A_1	20	30	20	12
A_2	10	30	25	30
A_3	30	5	15	10
A_4	20	10	20	10

3) Use WPM and WSM methods to solve the following decision matrix.

	C_1	C_2	C_3	C_4
Alts.	0.20	0.15	0.40	0.25
A_1	25	20	15	30
A_2	10	30	20	30
A_3	30	10	25	10